Optimal Partial Tiling of Manhattan Polyominoes

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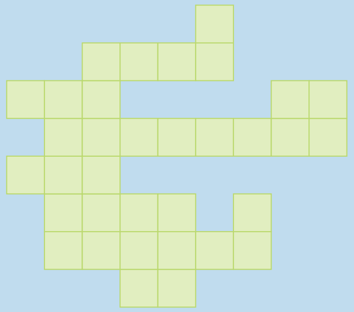
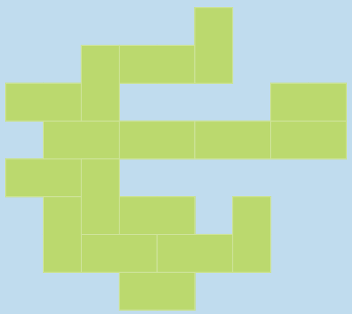
Context

Polyominoes were introduced by S. Golomb in the 60's, as the simply connected union of unit squares drawn on the $\mathbb{Z}^2$ lattice.

A natural question: can a given polyomino be *tiled* (that is, "covered") using dominoes?



In 1990, W. P. Thurston gave a *linear-time algorithm* to decide if a *polyomino without holes* can be tiled with dominoes [4]. For instance, here is one possible tiling of the above polyomino:

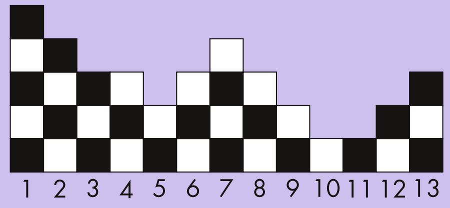


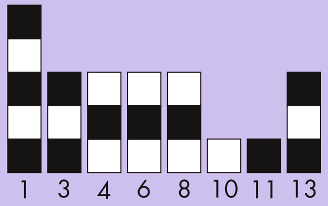
Similar results were extended to polyominoes without holes [1], but in all instances, algorithms have focused on *exact solutions*.

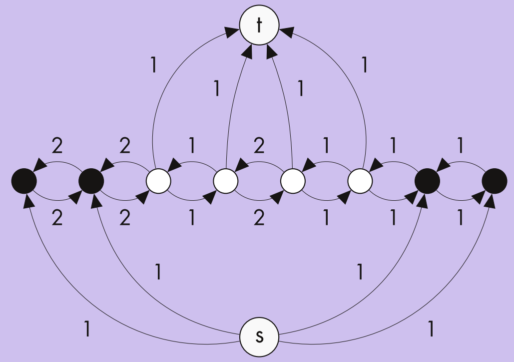
MAIN RESULT

Theorem. There is an algorithm which is linear in the size of the input (using an optimal coding), that gives the number of uncovered squares of an optimally-tiled Manhattan polyomino.

Principle. We transform Manhattan polyominoes P into flow networks F_P (the nodes of which are the oddly-sized columns), and these allow us to show our "greedy planing" technique is optimal.







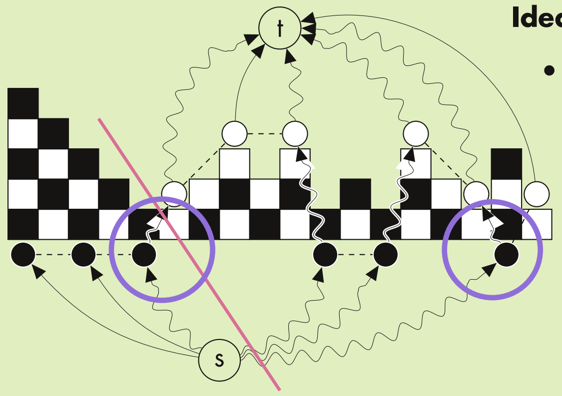
Proposition. Let P be a Manhattan polyomino, and $|I_P|$ and $|J_P|$ respectively be the number of oddly-sized columns which contain more black or white squares. Then the value $v(P)$ be the value of the maximum flow on F_P such that

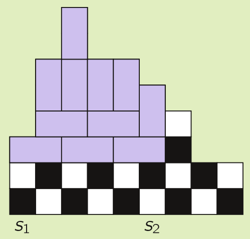
$$|I_P| + |J_P| - 2v(P) = d(P)$$

where $d(P)$ is the number of non-covered unit squares in an optimal partial tiling of P.

Idea. The proofs involve:

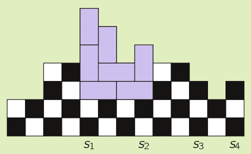
- Our planing transformation, a ready-made way of getting rid of *two* neighboring (only even columns in between) oddly-sized columns of different colors;





- The flow network translates how this planing transformation can be applied, and the maximum flow of this network corresponds to the maximum number of oddly-sized columns we can tile at the same time.

In this second step, the bottlenecks of the network (circled above) play a crucial role, as the planing transformation can be seen in the network as the saturation of a min-path.



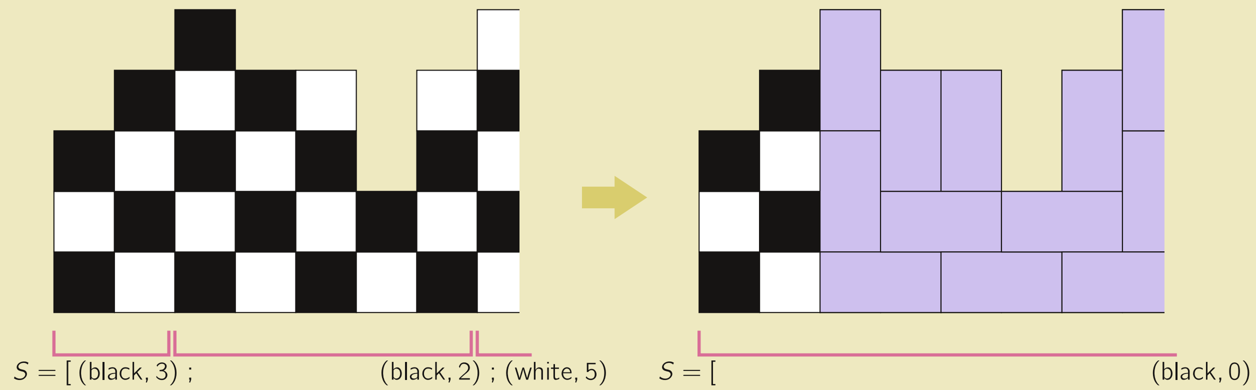
Algorithm 1: Greedy algorithm for the partial tiling of a Manhattan polyomino P, given as a list of heights of the columns.

```
S ← []
foreach column c of P do
    if c ≡ 1 mod 2 then              /* if the column is oddly-sized */
        Push(isBlack, c, S)
    else
        UpdateTop(c, S)
    AttemptCollapse(S)
    isBlack ← not isBlack           /* the colors alternate */
return Size(S)
```

S is a stack-like data structure, which *keeps track of the oddly-sized columns*, and whether it is *possible to apply the planing transformation*. All three "stack" operations are in $O(1)$.

The AttemptCollapse operation

This operation translates the planing transformation to the stack data structure.



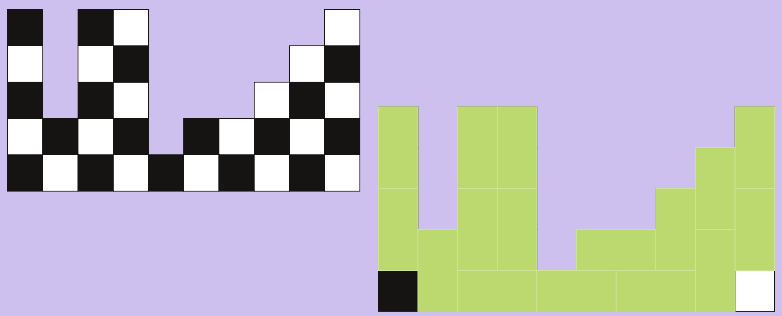
In the figure above, the algorithm has processed the 8 first columns of a polyomino: left, is *before* AttemptCollapse has been called; right, is *after*. The planing transformation was applied, and two oddly-sized columns were tiled (*their corresponding elements were popped from* S).

The second field of the S's remaining element is 0, because the minimum untiled-height from the remaining oddly-sized dominant column onwards is 0 (the relevant area is delimitated by the red segments).

Conjecture (and partial answer)

If a polyomino is not tileable, it is possible to determine, in *linear time*, the *maximum number of squares* that can be covered by non-overlapping dominoes.

The question remains open, but we have proved the subcase in which we restrict the class to *Manhattan polyominoes*.



References

[1] O. Bodini, T. Fernique, *Planar Dimer Tilings*, LNCS 3967, p. 104-113, Springer (2006).

[2] H. N. Gabow, R. E. Tarjan, *Faster scaling algorithms for network problems*, SIAM J. Comput., Vol. 18, #5, p. 1013-1036, (1989).

[3] J. E. Hopcroft, R. M. Karp, *An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs*, SIAM J. Comput., Vol. 2, #4, p. 225-231, (1973).

[4] W. P. Thurston, *Conway's tiling groups*, American Mathematics Monthly 95, p. 757-773, (1990).

[Oct. 1st 2009 - DGCI 2009 - Montreal]