Philippe Flajolet's

contribution to streaming algorithms

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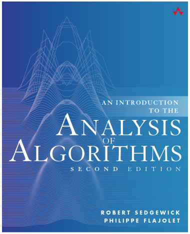
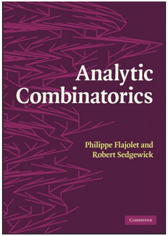
Philippe Flajolet (1948 - 2011)

- analysis of algorithms
  - worst-case analysis
  - 1970: Knuth, average case analysis
  - 1980: Rabin, introduce randomness in computations
- wide scientific production
  - two books with Robert Sedgewick
  - 200+ publications
- founder of the topic of "analytic combinatorics"
- published the first sketching/streaming algorithms





0. DATA STREAMING ALGORITHMS

Stream: a (very large) sequence S over (also very large) domain $\mathcal{D}$

$$S = s_1\ s_2\ s_3\ \cdots\ s_\ell, \qquad s_j \in \mathcal{D}$$

consider S as a multiset

$$\mathcal{M} = {m_1}^{f_1}\ {m_2}^{f_2}\ \cdots\ {m_n}^{f_n}$$

Interested in **estimating** the following *quantitive* statistics:

— **A.** **Length** $:= \ell$

— **B.** **Cardinality** $:= \mathrm{card}(m_i) \equiv n$ (distinct values) $\leftarrow$ this talk

— **C.** **Frequency moments** $:= \sum_{v \in \mathcal{D}} {f_v}^p \quad p \in \mathbb{R}_{\geqslant}$

Constraints:

- very little processing memory
- on the fly (single pass + simple main loop)
- no statistical hypothesis
- accuracy within a few percentiles

Historical context

- **1970**: average-case $\rightarrow$ deterministic algorithms on random input
- **1976-78**: first randomized algorithms (primality testing, matrix multiplication verification, find nearest neighbors)
- **1979**: Munro and Paterson, find median in one pass with $\Theta(\sqrt{n})$ space with high probability
  $\Rightarrow$ (almost) first streaming algorithm

In **1983**, Probabilistic Counting by Flajolet and Martin is (more or less) the first streaming algorithm (one pass + constant/logarithmic memory).

Google scholar

Probabilistic counting algorithms for data base applications
P **Flajolet**... - Journal of computer and system sciences, 1985 - Elsevier
Abstract This paper introduces a class of probabilistic counting algorithms with which one can estimate the number of distinct elements in a large collection of data (typically a large file stored on disk) in a single pass using only a small additional storage (typically less ...
Cited by 628 - Related articles - All 36 versions

Probabilistic counting
P **Flajolet**... - Foundations of Computer Science, ..., 1983 - ieeexplore.ieee.org
Abstract We present here a class of probabilistic algorithms with which one can estimate the number of distinct elements in a collection of data (typically a large file stored on disk) in a single pass, using only 0 (1) auxiliary storage and 0 (1) operations per element. We ...
Cited by 111 - Related articles - All 7 versions

Combining both versions: cited about 750 times = **second most cited** element of Philippe's bibliography, after only *Analytic Combinatorics*.

Databases, IBM, California...

In the 70s, IBM researches relational databases (first PRTV in UK, then System R in US) with high-level query language: user should not have to know about the structure of the data.

⇒ query optimization; requires cardinality (estimates)

```
SELECT name FROM participants
WHERE
   sex = "M" AND
   nationality = "France"
```

Min. comparisons: compare first `sex` or `nationality`?

G. Nigel N. Martin (IBM UK) invents first version of "probabilistic counting", and goes to IBM San Jose, in **1979**, to share with System R researchers. Philippe discovers the algorithm in **1981** at IBM San Jose.

As I said over the phone, I started working on your algorithm when Kyu-Young Whang considered implementing it and wanted explanations/estimations. I find it simple, eleg
and amazingly powerful.

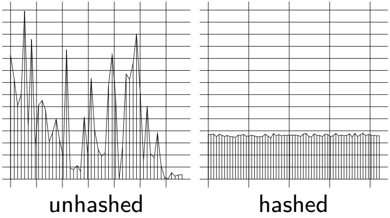
1. HASHING: reproducible randomness

- **1950s:** hash functions as tools for hash tables
- **1969:** Bloom filters $\rightarrow$ first time in an approximate context
- **1977/79:** Carter & Wegman, *Universal Hashing*, first time considered as probabilistic objects + proved uniformity is possible in practice

hash functions **transform data into i.i.d. uniform** random variables or in infinite strings of random bits:

$$h : \mathcal{D} \rightarrow \{0,1\}^{\infty}$$

that is, if $h(x) = b_1 b_2 \cdots$,
then $\mathbb{P}[b_1 = 1] = \mathbb{P}[b_2 = 1] = \ldots = 1/2$



- Philippe's approach was experimental
- later theoretically validated in **2010:** Mitzenmacher & Vadhan proved hash functions "work" because they exploit the entropy of the hashed data

2. PROBABILISTIC COUNTING (1983)

(with G. Nigel N. Martin)

For each element in the string, we hash it, and look at it

$$S = s_1 \; s_2 \; s_3 \; \cdots \qquad \Rightarrow \quad h(s_1) \; h(s_2) \; h(s_3) \; \cdots$$

$\boldsymbol{h(v)}$ **transforms** $\boldsymbol{v}$ **into string of random bits** (0 or 1 with prob. 1/2).
So you expect to see:

$$0xxxx \ldots \to \mathbb{P} = 1/2 \qquad 10xxx \ldots \to \mathbb{P} = 1/4 \qquad 110xx \ldots \to \mathbb{P} = 1/8$$

Indeed

$$\mathbb{P}\left[\ \boxed{1 \mid 1 \mid 0 \mid x \mid x \mid \cdots}\ \right] = \mathbb{P}[b_1 = 1] \cdot \mathbb{P}[b_2 = 1] \cdot \mathbb{P}[b_3 = 0] = \frac{1}{8}$$

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Intuition: because strings are uniform, prefix pattern $1^k0\cdots$ appears with probability $1/2^{k+1}$
$\Rightarrow$ seeing prefix $1^k0\cdots$ means it's likely there is $n \geqslant 2^{k+1}$ different strings

Idea:

- keep track of prefixes $1^k0\cdots$ that have appeared
- estimate cardinality with 2^p, where $p =$ size of largest prefix

Bias correction: how analysis is FULLY INVOLVED in design

Described idea works, but presents **small bias** (i.e. $\mathbb{E}[2^p] \neq n$).

Without analysis (original algorithm)

```
After all the values have been processed, then:

if M(MAP)=000,  then RESULT=LO(MAP)-1

if M(MAP)=111,  then RESULT=LO(MAP)+1

otherwise RESULT=LO(MAP).

For example,
   if MAP was 00000000000000000000000001111111
   LO(MAP) is 8 and M(MAP) is 000: RESULT=7
   if MAP was 00000000000000000000001110111111
   LO(MAP) is 8 and M(MAP) is 111: RESULT=9
   if MAP was 00000000000000000000001001111111
   LO(MAP) is 8 and M(MAP) is 010: RESULT=8
```

the three bits immediately after the first 0 are sampled, and depending on whether they are 000, 111, etc. a small ± 1 correction is applied to $p = \rho(\mathrm{bitmap})$

With analysis (Philippe)

Philippe determines that

$$\mathbb{E}[2^p] \approx \phi n$$

where $\phi \approx 0.77351\ldots$ is defined by

$$\phi = \frac{e^{\gamma}\sqrt{2}}{3} \prod_{p=1}^{\infty} \left[\frac{(4p+1)(4p+2)}{(4p)(4p+3)} \right]^{(-1)^{\nu(p)}}$$

such that we can apply a simple correction and have <u>unbiased</u> estimator,

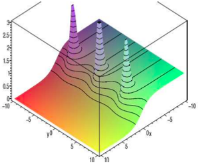
$$Z := \frac{1}{\phi} 2^p \qquad \mathbb{E}[Z] = n$$

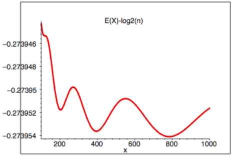
Analysis close-up: “Mellin transforms”

transformation of a function to the complex plane

$$f^\star(s) = \int_0^\infty f(x) x^{s-1} \mathrm{d}x.$$

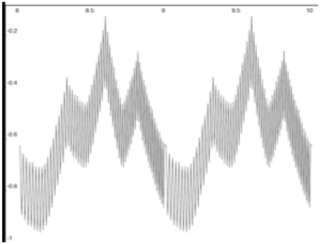
- factorizes linear superpositions of a base function at different scales
- links singularities in the complex plane of the integral, to asymptotics of the original function





precise analysis (better than “Master Theorem”) of all divide and conquer type algorithms (QuickSort, etc.) with recurrences such as

$$f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + t_n$$



$v_1(n)$ is # of 1's in binary representation of n.

$H(n) = \sum_{i<n} v_1(n)$ is an interesting arithmetic function.

$$H(n) = \frac{1}{2} n \lg n + nD(\lg n)$$

(graphic: M. Golin)

The basic algorithm

- $h(x)$ = hash function, transform data x into uniform $\{0,1\}^\infty$ string
- $\rho(s)$ = position of first bit equal to 0, i.e. $\rho(1^k 0 \cdots) = k+1$

procedure ProbabilisticCounting(S : stream)
    $\mathrm{bitmap} := [0, 0, \ldots, 0]$
    for all $x \in S$ **do**
        $\mathrm{bitmap}[\rho(h(x))] := 1$
    end for
    $P := \rho(\mathrm{bitmap})$
    return $\frac{1}{\phi} \cdot 2^P$
end procedure

Ex.: if $\mathrm{bitmap} = 1111000100 \cdots$ then $P = 5$, and $n \approx 2^5/\phi = 20.68\ldots$

Typically estimates are one binary order of magnitude off the exact result: **too inaccurate for practical applications**.

Stochastic Averaging

To improve accuracy of algorithm by $1/\sqrt{m}$, elementary idea is to use m different hash functions (and a different bitmap table for each function) and **take average**.

$\Rightarrow$ very costly (hash m time more values)!

Split elements in m substreams randomly using first few bits of hash

$$h(v) = b_1 b_2 b_3 b_4 b_5 b_6 \cdots$$

which are then discarded (only $b_3 b_4 b_5 \cdots$ is used as hash value).

For instance for $m = 4$,

$$h(x) = \begin{cases} 00 b_3 b_4 \cdots & \to & \mathrm{bitmap}_{00}[\rho(b_3 b_4 \cdots)] = 1 \\ 01 b_3 b_4 \cdots & \to & \mathrm{bitmap}_{01}[\rho(b_3 b_4 \cdots)] = 1 \\ 10 b_3 b_4 \cdots & \to & \mathrm{bitmap}_{10}[\rho(b_3 b_4 \cdots)] = 1 \\ 11 b_3 b_4 \cdots & \to & \mathrm{bitmap}_{11}[\rho(b_3 b_4 \cdots)] = 1 \end{cases}$$

Theorem [FM85]. The estimator Z of Probabilistic Counting is an **asymptotically unbiased** estimator of cardinality, in the sense that

$$\mathbb{E}_n[Z] \sim n$$

and has accuracy using m bitmaps is

$$\frac{\sigma_n[Z]}{n} = \frac{0.78}{\sqrt{m}}$$

Concretely, need $O(m \log n)$ memory (instead of $O(n)$ for exact).

Example: can count cardinalities up to $n = 10^9$ with error $\pm 6\%$, using only 4096 bytes = 4 kB.

3. from Prob. Count. to LogLog (2003)

(with Marianne Durand)

PC: bitmaps require k bits to count cardinalities up to $n = 2^k$

Reasoning backwards (from observations), it is reasonable, when estimating cardinality $n = 2^3$, to observe a bitmap $11100\cdots$; remember

- $b_1 = 1$ means $n \geqslant 2$
- $b_2 = 1$ means $n \geqslant 4$
- $b_3 = 1$ means $n \geqslant 8$

WHAT IF instead of keeping track of **all** the 1s we set in the bitmap, we only kept track of the **position of the largest**? It only requires $\log \log n$ bits!

In algorithm, replace

$$\mathrm{bitmap}_i[\rho(h(x))] := 1 \qquad \text{by} \qquad \mathrm{bitmap}_i := \max\{\rho(h(x)), \mathrm{bitmap}_i\}$$

For example, compared evolution of "bitmap":

Prob. Count.:	$00000\cdots$	$00100\cdots$	$10100\cdots$	$11100\cdots$	$11110\cdots$
LogLog:	1	4	4	4	5

loss of precision in LogLog?

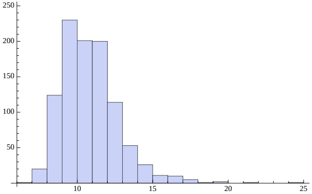
Probabilistic Counting and LogLog **often** find the same estimate:

Probabilistic Counting					**5**				
LogLog					**5**				
bitmap	1	1	1	1	0	0	0	0	$\cdots$

But sometimes differ:

Probabilistic Counting					**5**				
LogLog								**8**	
bitmap	1	1	1	1	0	0	1	0	$\cdots$

Other way of looking at it, the **distribution** of the rank (= max of n geometric variables with $p = 1/2$) used by LogLog has **long tails**:



(still there is concentration: idea of compressing the sketches, e.g. optimum by Kane et al. 2000)

SuperLogLog (same paper)

The accuracy (want it to be smallest possible):

- **Probabilistic Counting:** $0.78/\sqrt{m}$ for m registers of 32 bits
- **LogLog:** $1.36/\sqrt{m}$ for m small registers of 5 bits

In LogLog, loss of accuracy due to some (rare but real) registers that are too big, too far beyond the expected value.

SuperLogLog is LogLog, in which we remove δ largest registers before estimating, i.e., $\delta = 70\%$.

- involves a two-time estimation
- analysis is much more complicated
- but accuracy much better: $\mathbf{1.05/\sqrt{m}}$

Analysis — Duper Loglog — Nov 1, 2006

Geometric RV.

$$\mathbb{P}(X=k)=\frac{1}{2^k} \qquad k=1,2,3,\ldots$$

$$\mathbb{P}(X\geq k)=\frac{1}{2^{k-1}} \qquad k=1,2,3,\ldots$$

$$\mathbb{P}(X<k)=\frac{1}{2^{k-1}}\,;\qquad \mathbb{P}(X\leq k)=1-1/2^k.$$

Max Geom $\quad M_\nu=\max\left(X^{(1)},\ldots,X^{(\nu)}\right) \qquad X^{(j)}\in \text{geom}\left(\frac{1}{2}\right)$

$$\mathbb{P}(M_\nu\leq k)=\left(1-\frac{1}{2^k}\right)^\nu \quad \begin{cases} \text{valid for } \nu>0,\ k=0,1,2,3,\ldots \\ \text{or } \nu=0 \text{ with convention } 0^0=1 \\ [\max(\{\phi\})=0.] \end{cases}$$

Normalizing : $\boxed{\text{Correct bias by considering } \alpha=2\log 2}$

Let $S=M_\nu^{(1)}+\cdots+M_\nu^{(m)}$, the sum of m independent copies

$$\boxed{\begin{array}{l}\mathbb{E}\left(\frac{S}{m}\times 2\log 2\right)\approx x^{-1}. \qquad \mathrm{Var}\left(\frac{S}{m}\times 2\log 2\right)\approx\frac{x^{-2}}{m}(3\log 2-1) \\ \text{Set } \beta=\sqrt{3\log 2-1},\ \beta=1.03896. \qquad (5)\end{array}}$$

4. $\boxed{\text{“HyperLogLog:}}$

the analysis of a near-optimal cardinality estimation algorithm” (2007)

(with Eric Fusy, Frédéric Meunier & Olivier Gandouet)

- **2005:** Giroire (PhD student of Philippe’s) publishes thesis with cardinality estimator based on order statistics
- **2006:** Chassaing and Gerin, using statistical tools find best estimator based on order statistics in an information theoretic sense

The note suggests using a harmonic mean: initially dismissed as a theoretical improvement, it turns out simulations are *very* good. Why?

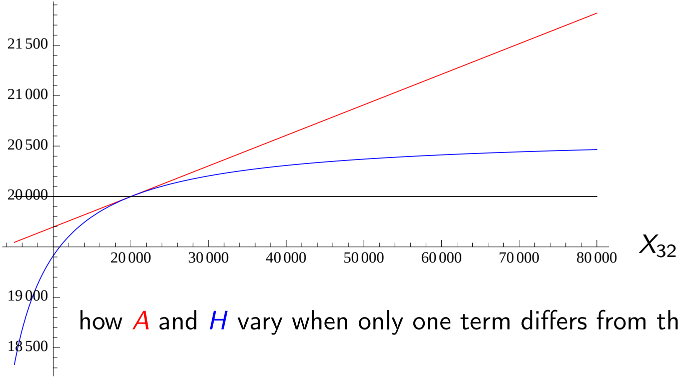
Harmonic means ignore too large values

X_1, X_2, ..., X_m are estimates of a stream's cardinality

Arithmetic mean **Harmonic mean**

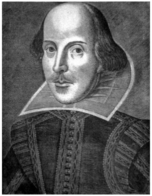
$$A := \frac{X_1 + X_2 + \ldots + X_m}{m} \qquad H := \frac{m}{\frac{1}{X_1} + \frac{1}{X_2} + \ldots + \frac{1}{X_m}}$$

Plot of A and H for $X_1 = \ldots = X_{31} = 20\,000$ and X_{32} varying between and 5 000 and 80 000 (two binary orders of magnitude)



how A and H vary when only one term differs from the rest

The end of an adventure. HyperLogLog = sensibly same precision as SuperLogLog, but **substitutes** algorithmic cleverness with **mathematical elegance**.



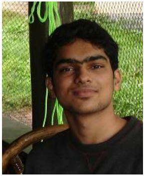
Accuracy is $\mathbf{1.03/\sqrt{m}}$ with m small loglog bytes (≈ 4 bits).

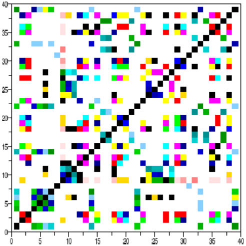
Whole of Shakespeare summarized:

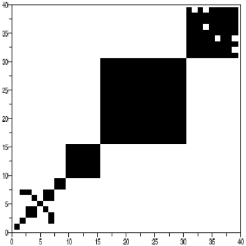
```
ghfffghfghgghggggghghheehfhfhhgghghghhfgffffhhhiigfhhffgfiihfhhh
igigighfgihfffghigihghigfhhgeegeghgghhhgghhfhidiigihighihehhhfgg
hfgighigffghdieghhhggghhfghhfiiheffghghihifgggffihgihfggighgiiif
fjgfgjhhjiifhjgehgghfhhfhjhigggghghihigghhihihgiighgfhlgjfgjjjmfl
```

Estimate $\tilde{n} \approx 30\,897$ against $n = 28\,239$. Error is $\pm 9.4\%$ for 128 bytes.

Pranav Kashyap: word-level encrypted texts, classification by language.







Left out of discussion:

- Philippe's finding and analysing of Approximate Counting, 1982: how to count up to n with only $\log \log n$ memory

Left out of discussion:

- Philippe's finding and analysing of Approximate Counting, 1982: how to count up to n with only $\log \log n$ memory
- a beautiful algorithm (with Wegman), Adaptive Sampling, 1989, which was ahead of its time, and was grossly unappreciated... until it was rediscovered in 2000: how do you count the number of elements which appear only once in a stream using constant size memory?

A. $\boxed{\text{adaptive/DISTINCT sampling}}$

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Let S be a stream of size ℓ (with n distinct elements)

$$\mathcal{S} = x_1 \; x_2 \; x_3 \; \cdots \; x_\ell$$

- a **straight sample** [Vitter 85..] of size m (each x_i taken with prob. $\approx m/\ell$)

  `a x x x x b b x c d d d b h x x ...`

  allows us to deduce 'a' repeated $\approx \ell/m$ times in S, but impossible to say anything about rare elements, hidden in the mass = **problem** of needle in haystack
- a **distinct sample** (with counters)

  `(a, 9) (x, 134) (b, 25) (c, 12) (d, 30) (g, 1) (h, 11)`

  takes each element with probability $1/n$ = **independently** from its **frequency** of appearing

Textbook example: sample 1 element of stream $(1, 1, 1, 1, 2, 1, 1, \ldots, 1)$, $\ell = 1000$; with straight sampling, prob. $999/1000$ of taking 1 and $1/1000$ of taking 2; with distinct sampling, prob. $1/2$ of taking 1 and $1/2$ of taking 2.