An Exact Enumeration of Distance-Hereditary Graphs

Jérémie Lumbroso (Princeton)
Joint work with Cédric Chauve (SFU)
and Éric Fusy (LIX, Polytechnique).

ALENEX/ANALCO/SODA 2017, Barcélona
Jan. 16-19th, 2017

0. Motivation and Outline

Motivation:

- in this talk: precisely enumerate large classes of graphs
- we combine in novel way:
  - classical characterization of graphs by tree-decompositions—because trees are **easier to count**
  - "graph labeled tree" framework (Gioan and Paul, 2012)
  - techniques in analytic combinatorics (symbolic method + asymptotic theorems)
  - technique from species theory (dissymetry theorem on trees)
- obtain exact and asymptotic enumerations + more

Outline:

- present definitions (graph decomposition, split decomposition, symbolic method)
- illustrate our approach for a simpler class of graphs (3-leaf power graphs)
- results for distance-hereditary graphs
- perspectives

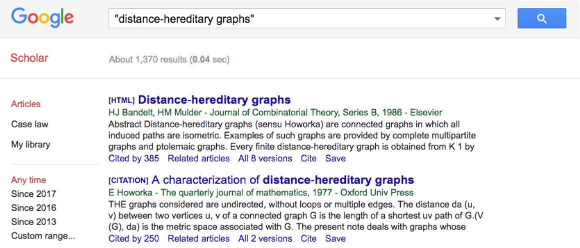
context: some direct predecessors of our method

this work is informed by a long line of research on graph decomposition (see Gioan and Paul especially), but two prior works are particular relevant:

- **Thimonier and Ravelomanana 2002:** asymptotic enumeration of cographs (totally decomposable graphs for modular decomposition) using analytic combinatorics techniques
- **Nakano *et al.* 2007:** encoding and upper-bound for enumeration of distance-hereditary graphs (totally decomposable graphs for split decomposition) using algorithmic construction
- **Gioan and Paul, 2009-2012:** introduced the notion of graph-labeled tree and way to characterize split-decomposition output

goal: develop general methods cover vast subsets of **perfect graphs**[1]

starting point distance-hereditary graphs: [all as of Jan. 16th]



- planar graphs: 44 500 results
- interval graphs: 11 600 results [imperfect: incl. in perf. gr.]
- perfect graphs: 9 990 result
- chordal graphs: 8 860 results
- series-parallel graphs: 4 720 results
- cographs: 2 690 results
- block graphs: 1940 results

[1] chromatic number of every induced subgraph = size of max-clique of subgraph

context: distance-hereditary graphs (2)

- **1977, Howorka:** defines DH graphs (respect isometric distance: all induced paths between two vertices are same length)
- **1982, Cunningham:** introduces split-decomposition (as "join decomposition")
- **1986, Bandelt and Mulder:** vertex-incremental characterization
- **1990, Hammer and Maffray:** DH graphs are totally decomposable by the split-decomposition
- **2003, Spinrad:** upper-bound of enumeration sequence $2^{O(n \log n)}$
- **2009, Nakano *et al.*:** upper-bound of $2^{\lceil 3.59n \rceil}$ (approx. within factor 2)
- **2014-16, Chauve, Fusy, L.:** exact enumeration + full asymptotic (= constant, polynomial and exp. terms)

1. Graph decompositions

Def: a *graph-labeled tree* (GLT) is a pair $(T, \mathcal{F})$, with T a tree and $\mathcal{F}$ a set of graphs such that:

- a node v of degree k of T is labeled by graph $G_v \in \mathcal{F}$ on k vertices;
- there is a bijection ρ_v from the tree-edges incident to v to the vertices of G_v.

1. Graph decompositions

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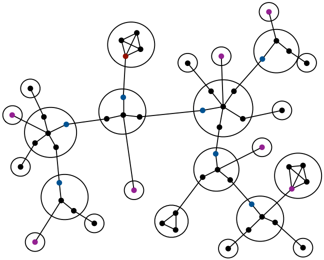
- there is a bijection ρ_v from the tree-edges incident to v to the vertices of G_v.

Def: a *rooted graph-labeled tree* is a graph-labeled tree of which one internal node is distinguished.

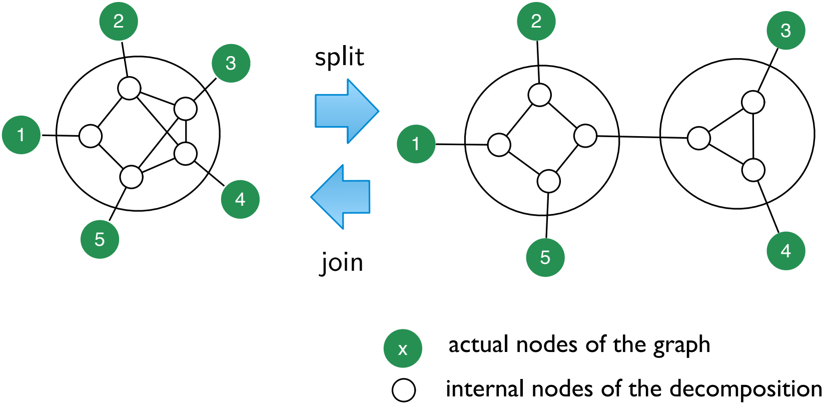
Remark: several types of decompositions of graphs (modular, split...); each decomposition has **totally decomposition graphs** for which the decomposition does not contain internal prime graphs.



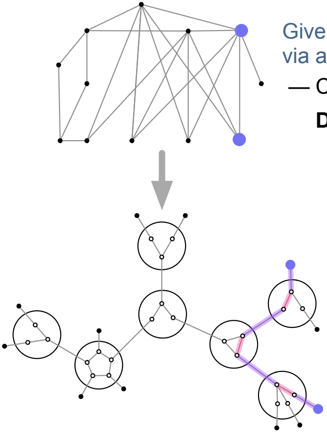
split decomposition (1)

Def: a bipartition (A, B) of a the vertices of a graph is a *split* iff

- $|A| \geqslant 2$, $|B| \geqslant 2$;
- for $x \in A$ and $y \in B$, $xy \in E$ iff $x \in N(B)$ and $y \in N(A)$.



split decomposition (2)

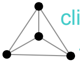


Gives a *graph-labeled tree* representation of a graph via a series of *split* operations

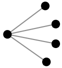
— Can read adjacencies from *alternated paths*.

Decomposition base cases:

degenerate nodes**:**



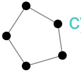
clique $\mathcal{K}$



star $\mathcal{S}$

prime nodes:

e.g.



cycle $\mathcal{P}$

Theorem (Cunningham '82):
The split decomposition tree into *prime* and *degenerate* nodes is unique as long as certain conditions are met.

Theorem:
Cycles of size at least 5 are prime nodes.

Remark:

- *distance-hereditary graphs*: graphs that are totally decomposable by split decomposition: internal nodes are star-nodes or clique-nodes;
- *3-leaf power graphs*: **subset** of distance-hereditary graphs, with additional constraint that star nodes form connected subtree.

2. Specifiable Combinatorial Classes

a class $\mathcal{A}$ is a specifiable combinatorial class if:

- described by symbolic rules (= grammar)

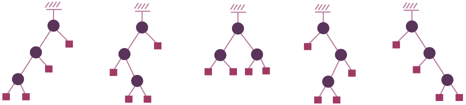
$\mathcal{Z}$, ε	$+$, $\times$, Seq, Set, Cyc, …
building blocks	ways to combine them

- possible recursive (defined using itself)
- the number a_n of objects of size n is finite

Example: class $\mathcal{B}$ of binary trees specified by

$$\mathcal{B} = \varepsilon + \mathcal{B} \times \mathcal{Z} \times \mathcal{B}$$

all binary trees of size 3 (with 3 internal nodes $\bullet$):



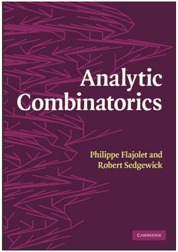
symbolic method [Flajolet & Sedgewick 09]

the **generating function** $A(z)$ of class $\mathcal{A}$ encodes, within a function, the complete enumeration (the number of objects for each size) of the class:

$$A(z) = \sum_{n=0}^{\infty} a_n z^n$$

- in the general case, this generating function (GF) is a formal object; however the GF of decomposable classes is often convergent
- dictionary: correspondence which exactly relates specific. and GF

construction	specification	GF
neutral element	ε	1
atome	$\mathcal{Z}$	z
union	$\mathcal{A} + \mathcal{B}$	$A(z) + B(z)$
Cartesian product	$\mathcal{A} \times \mathcal{B}$	$A(z) \cdot B(z)$
sequence	$\mathrm{Seq}(\mathcal{A})$	$\frac{1}{1-A(z)}$



example: class $\mathcal{B}$ of binary trees

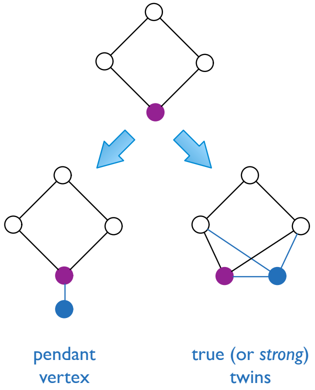
$$\mathcal{B} = \varepsilon + \mathcal{B} \times \mathcal{Z} \times \mathcal{B} \quad \Rightarrow \quad B(z) = 1 + B(z) \cdot z \cdot B(z) = \frac{1 - \sqrt{1-4z}}{2}$$

3. 3-LEAF POWER graphs

(One Possible) Def: a connected graph is a 3-leaf power graphs (3LP) iff it results from a tree by replacing every vertex by a clique of arbitrary size.

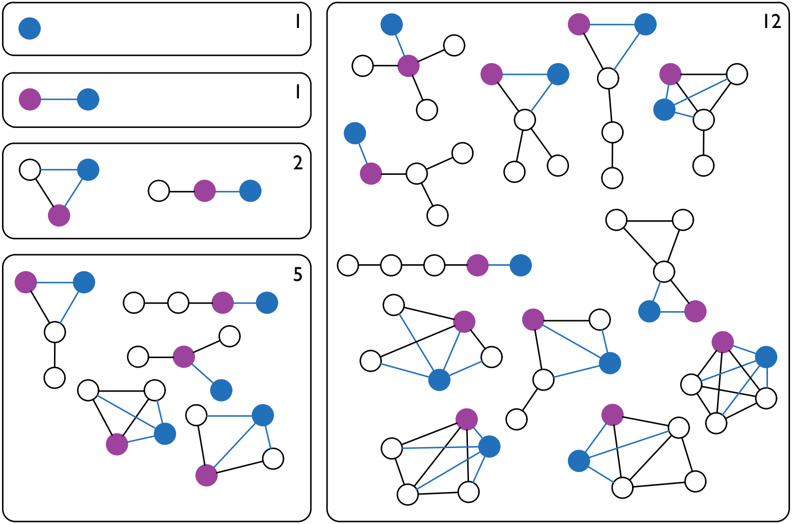
Algorithmic Characterization: 3LP graphs are obtained from a single vertex by

- **first** iterating arbitrary additions of pendant vertex;
- **then** iterating arbitrary additions of true twins.



(This caracterization is especially useful when establishing a reference, brute-force enumeration of these graphs!)

the first few 3-leaf power graphs



if these graphs were to be constructed by incremental construction, the **blue vertex** represents the vertex added from a smaller graph

obtaining **rooted** grammar of 3LP

Split-tree characterization of 3LP graphs:

1. its split tree $\mathrm{ST}(G)$ has only of clique-nodes and star-nodes;
2. the set of star-nodes forms a connected subtree of $\mathrm{ST}(G)$;
3. the center of a star-node is incident either to a leaf or a clique-node.

From this, we describe **rooted** tree decomposition, by walking through the tree

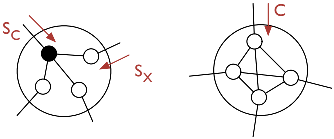
$$3\mathcal{LP}_\bullet = \mathcal{L}_\bullet \times (\mathcal{S}_C + \mathcal{S}_X) + \mathcal{C}_\bullet \qquad \mathcal{S}_C = \mathsf{Set}_{\geq 2}(\mathcal{L} + \mathcal{S}_X)$$

$$\mathcal{S}_X = \mathcal{L} \times \mathsf{Set}_{\geqslant 1}(\mathcal{L} + \mathcal{S}_X) \qquad \mathcal{L} = \mathcal{Z} + \mathsf{Set}_{\geqslant 2}(\mathcal{Z})$$

$$\mathcal{L}_\bullet = \mathcal{Z}_\bullet + \mathcal{Z}_\bullet \times \mathsf{Set}_{\geqslant 1}(\mathcal{Z}) \qquad \mathcal{C}_\bullet = \mathcal{Z}_\bullet \times \mathsf{Set}_{\geq 2}(\mathcal{Z})$$

where

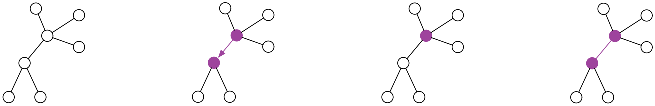
- $\mathcal{S}_C$ are star-nodes entered through their center; $\mathcal{S}_X$, their extremities;
- $\mathcal{A}_\bullet$ is a class where one vertex is distinguished;
- $\mathcal{L}$ are leaves (either cliques or single vertices) and $\mathcal{C}$ (clique).



from rooted to **unrooted**: dissymetry theorem for trees

- the grammars obtained describe a class of **rooted** trees; so the identical graphs are counted several times
- we need a tool to transform these grammars into grammars for the **equivalent unrooted class**;
- one such tool, the **Dissymetry Theorem for Trees** [Bergeron *et al.* 98] states

$$\mathcal{A} \quad + \quad \mathcal{A}_{\circ\to\circ} \quad \simeq \quad \mathcal{A}_{\circ} \quad + \quad \mathcal{A}_{\circ-\circ}$$



  with

  - $\mathcal{A}$, unrooted class (which we are looking for)
  - $\mathcal{A}_\circ$, class rooted **node** (which we have)
  - $\mathcal{A}_{\circ-\circ}$ and $\mathcal{A}_{\circ\to\circ}$, class respectively rooted in **undirected edge** and **directed edge** (easy to obtain from $\mathcal{A}_\circ$)
- alternate tool: cycle pointing (more difficult but preserves combinatorial grammar)

unrooted grammar — just for your information

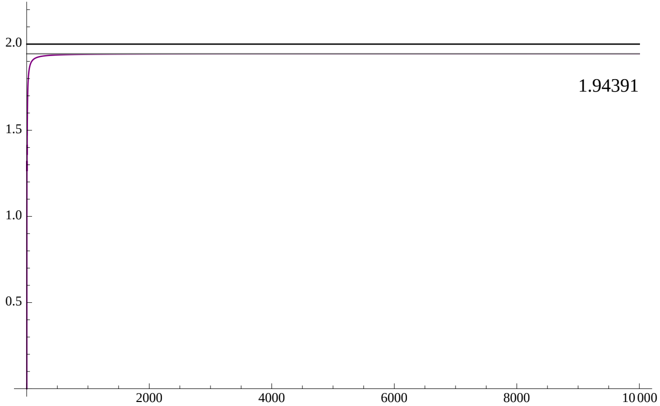
- from dissymetry theorem, we deduce $\mathcal{A} = \mathcal{A}_\circ + \mathcal{A}_{\circ-\circ} - \mathcal{A}_{\circ\to\circ}$ *for the purposes of enumeration*
- thus the *unrooted* 3LP graphs are described by

$$
\begin{aligned}
3\mathcal{LP} &= \mathcal{C} + \mathcal{T}_S + \mathcal{T}_{S-S} - \mathcal{T}_{S\to S} \\
\mathcal{T}_S &= \mathcal{L} \times \mathcal{S}_C \\
\mathcal{T}_{S-S} &= \mathrm{Set}_2\left(\mathcal{S}_X\right) \\
\mathcal{T}_{S\to S} &= \mathcal{S}_X \times \mathcal{S}_X \\
\mathcal{S}_C &= \mathrm{Set}_{\geqslant 2}\left(\mathcal{L} + \mathcal{S}_X\right) \\
\mathcal{S}_X &= \mathcal{L} \times \mathrm{Set}_{\geqslant 1}\left(\mathcal{L} + \mathcal{S}_X\right) \\
\mathcal{L} &= \mathcal{Z} + \mathrm{Set}_{\geqslant 2}\left(\mathcal{Z}\right) \\
\mathcal{C} &= \mathrm{Set}_{\geqslant 3}\left(\mathcal{Z}\right).
\end{aligned}
$$

- remark:
  - original terms: $\mathcal{S}_C$, $\mathcal{S}_X$, $\mathcal{L}$, $\mathcal{C}$
  - terms from the dissymetry theorem: $\mathcal{T}_S$, $\mathcal{T}_{S-S}$, $\mathcal{T}_{S\to S}$
  - main term in the form of $\mathcal{A} = \mathcal{A}_\circ + \mathcal{A}_{\circ-\circ} - \mathcal{A}_{\circ\to\circ}$

experimental enumerations for graphs of size up to 10 000 (1)

- t_n: # of unlabeled and unrooted 3LP graphs of size n
- we know that $t_n = O(\alpha^n)$, want to find α
- here, plot of $\log_2(t_n/t_{n-1})$
- suggests growths of $\alpha = 2^{1.943\ldots}$ for 3-Leaf Power Graphs



experimental enumerations for graphs of size up to 10 000 (2)

Maple code to obtain previous plot, which allows to conjecture the asymptotic enumeration, once a grammar for the trees is found.

```
with(combstruct): with(plots):
TLP_UNROOTED_PARTS := {
  z = Atom,
  G_SUPERSET = Union(C, Union(TS, TSSu)),
  TS         = Prod(L, SC),
  TSSu       = Set(SX, card=2),
  TSSd       = Prod(SX, SX),
  SC         = Set(Union(L, SX), card >= 2),
  SX         = Prod(L, Set(Union(L, SX), card >= 1)),
  L          = Union(z, Set(z, card >= 2)),
  C          = Set(z, card >= 3)
}:
N := 10000:
OGF_TLP_SUPERSET := add(count([G_SUPERSET, TLP_UNROOTED_PARTS, unlabeled],
                              size = n) * x^n, n = 1 .. N):
OGF_TLP_TSSd := add(count([TSSd, TLP_UNROOTED_PARTS, unlabeled], size = n) *
                    x^n, n = 1 .. N):
OGF_TLP := OGF_TLP_SUPERSET - OGF_TLP_TSSd:
TLP_RATIOS := [seq([i, evalf(log(coeff(OGF_TLP, x, i)/coeff(OGF_TLP, x, i-1)))
                             /log(2)], i = 10 .. N)]:
plot(TLP_LOGS);
```

- the asymptotics of a algebraic grammar (described only with $+$ and $\times$, not sets) is well-known under theorem of Drmota-Lalley-Woods
- usually extends with no problem to other operations, under some niceness hypotheses [for ex., Chapuy *et al.* 08]

Method (without correctness proof):

1. let combinatorial system $\mathcal{S}$

$$\mathcal{S}\begin{cases} \mathcal{X}_1 & = & \Phi_1(\mathcal{X}_1, \ldots, \mathcal{X}_m) \\ \vdots & \vdots & \vdots \\ \mathcal{X}_m & = & \Phi_m(\mathcal{X}_1, \ldots, \mathcal{X}_m) \end{cases}$$

2. translate to equations on generating functions

$$\begin{array}{ccc} 0 & = & -X_1(z) + \phi_1(X_1(z), \ldots, X_m(z), z) \\ \vdots & \vdots & \vdots \\ 0 & = & -X_m(z) + \phi_m(X_1(z), \ldots, X_m(z), z) \end{array}$$

   with additional equation for recursion well-foundness

$$0 = \det(\mathrm{Jacobian}(\mathcal{S}))$$

3. solve numerically

Practical tweaks:

- our grammars involve unlabeled set operations, which result in infinite Polya series: these must be truncated
- additionally, singularity (= inverse of exponential growth) of rooted and unrooted classes is same: so work on (simpler) rooted grammar

Result: implemented algorithm in Maple, to obtain asymptotic of graph-decomposition with **arbitrary precision**:

```
TLP_ROOTED := {
  Gp = Union(Prod(Lp, Union(SC, SX)), Cp),
  SC = Set(Union(L, SX), card >= 2),
  SX = Prod(L, Set(Union(L, SX), card >= 1)),
  Cp = Prod(v, Set(v, card >= 2)), v = Atom, # [... snipped ...]
}:
fsolve_combsys(TLP_ROOTED, 100, z);
```

Eq1 = 0.02370404136, Eq2 = 0.5329652240, Eq3 = 0.3510690027,
Eq4 = 0.3510690027, Eq5 = 0.8016703909, Eq6 = 0.6489309973,
Eq7 = 0.2598453536, **z = 0.2598453536**

asymptotic exponential growth = $1/z$

4. Summary

We have used the example of 3-Leaf Power Graphs, because it is simpler to present, but **all results obtained for Distance-Hereditary graphs**.

Exact and asymptotic results for two major classes, previously unknown.

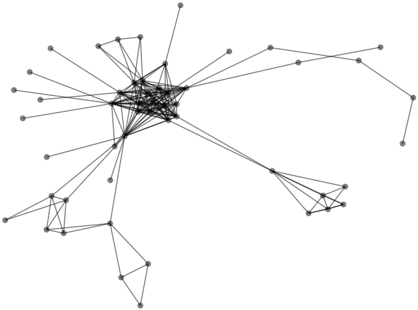
3-Leaf Power Graphs:

- exact enumeration: 1, 1, 2, 5, 12, 32, 82, 227, 629, 1840, 5456, 16701, 51939, 164688, ... (calculated linearly as function of size n)
- asymptotics: $c \cdot 3.848442876\ldots^n \cdot n^{-5/2}$ with $c \approx 0.70955825396\ldots$ (bound: $2^{1.9442748333}$)

Distance-Hereditary Graphs:

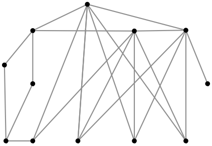
- exact enumeration: 1, 1, 2, 6, 18, 73, 308, 1484, 7492, 40010, 220676, 1253940, ... (calculated linearly as function of size n)
- asymptotics: $c \cdot 7.249751250\ldots^n \cdot n^{-5/2}$ with $c \approx 0.02337516194\ldots$ (bound: $2^{2.857931495}$)

random DH of size 52 [Iriza 15]

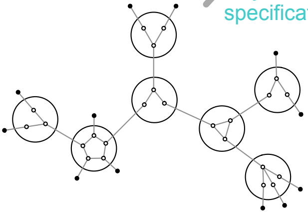


$$c \cdot 7.249751250\ldots^{n} \cdot n^{-5/2}$$

asymptotic theorems



split decomposition



symbolic specification

$$\mathcal{G} = \mathcal{Z} \times (\mathcal{P} + \mathcal{S}_C)$$
$$\mathcal{P} = \textsc{Seq}_{=4}(\mathcal{Z} + \mathcal{S}_X)$$
$$\mathcal{S}_X = \mathcal{Z} \times \textsc{Seq}_{\geqslant 1}(\mathcal{P})$$
$$\mathcal{S}_C = \textsc{Cyc}_{\geqslant 2}(\mathcal{P})$$

computer algebra system (CAS)

0, 0, 1, 0, 1, 0, 2, 0, 4, 0, 8, 0, 19, 0, 48, 0, 126, 0, 355, 0, 1037, ...

An Exact Enumeration of Distance-Hereditary Graphs

Cédric Chauve* Éric Fusy† Jérémie Lumbroso‡

Abstract

Distance-hereditary graphs form an important class graphs, from the theoretical point of view, due to fact that they are the totally decomposable graphs for

Theorem 4. *The class $\mathcal{DH}$ of unrooted distance-hereditary graphs is specified by*

$$\mathcal{DH} = \mathcal{T}_K + \mathcal{T}_S + \mathcal{T}_{S-S} - \mathcal{T}_{K-S} - \mathcal{T}_{S\to S} \quad (3.25)$$
$$\mathcal{T}_K = \textsc{Set}_{\geqslant 3}(\mathcal{Z} + \mathcal{S}_C + \mathcal{S}_X) \quad (3.26)$$
$$\mathcal{T}_S = (\mathcal{Z} + \mathcal{K} + \mathcal{S}_C) \times \mathcal{S}_C \quad (3.27)$$
$$\mathcal{T}_{K-S} = \mathcal{K} \times (\mathcal{S}_C + \mathcal{S}_X) \quad (3.28)$$
$$\mathcal{T}_{S-S} = \textsc{Set}_2(\mathcal{S}_C) + \textsc{Set}_2(\mathcal{S}_X) \quad (3.29)$$
$$\mathcal{T}_{S\to S} = \mathcal{S}_C \times \mathcal{S}_C + \mathcal{S}_X \times \mathcal{S}_X \quad (3.30)$$
$$\mathcal{K} = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_C + \mathcal{S}_X) \quad (3.31)$$
$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \quad (3.32)$$
$$\mathcal{S}_X = \textsc{Seq}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_C). \quad (3.33)$$

Enumerations, Forbidden Subgraph Characterizations, and the Split-Decomposition

Maryam Bahrani* Jérémie Lumbroso*

Abstract

Forbidden characterizations may sometimes be the most natural way to describe families of graphs, and yet these characterizations are usually very hard to exploit for enumerative purposes.

By building on the work of Gioan and Paul (2012) and Chauve et al. (2014), we show a methodology by which we constrain a split-decomposition tree to avoid certain patterns, thereby avoiding the corresponding induced subgraphs in the

As far as we know, while these notions are part and parcel of the work of graph theorists, they are usually not exploited by analytic combinatorists. For forbidden minors, there is the penetrating article of Bousquet-Mélou and Weller [4]. For forbidden subgraphs or forbidden induced subgraphs, we know of few papers, except because of the simple nature of graphs [31], or because some other, alternate property is used instead [5], or only asymptotics are determined [32].

We are concerned, in this paper, with *forbidden induced*

n induced subgraphs. ...ively well-known graph ...*omposition*, could be a ...in class called *distance-* ...meration had until then ...wn result was the bound ...ed that there are at most ...ry graphs on *n* vertices). ... version of this split- ... and Gioan, with inter- ...ved the legibility of the

Theorem 5. *The class $\mathcal{PG}_\bullet$ of ptolemaic graphs rooted at a vertex is specified by*

$$\mathcal{PG}_\bullet = \mathcal{Z}_\bullet \times (\mathcal{S}_C + \mathcal{S}_X + \mathcal{K}) \quad (4.15)$$
$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \quad (4.16)$$
$$\mathcal{S}_X = (\mathcal{Z} + \overline{\mathcal{K}}) \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \quad (4.17)$$
$$\mathcal{K} = \mathcal{S}_C \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{S}_X) + \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) \quad (4.18)$$
$$\overline{\mathcal{K}} = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) \quad (4.19)$$

5. Perspectives and upcoming results

Analyses:

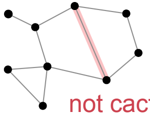
- **Parameter analysis:** analyzing, either theoretically or experimentally (already possible using random generation) various parameters of these graphs; such as distribution of star-nodes, clique-nodes, etc.
- **Other classes:** extending methodology to non-totally decomposable classes of graphs—either for modular decomposition or split decomposition (challenge is characterizing prime graphs in grammars).
  - bounds on parity graphs with bipartite prime [Shi, 2016 + ongoing]
  - forbidden subgraph characterizations [Bahrani and L., 2016]
  - cactus graphs [Bahrani and L., 2017]

Applications:

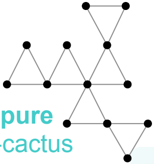
- **Encoding:** asymptotic result suggests more efficient encoding than the one provided by Nakano *et al.* 2007 (which uses 2^{4n} bits)?
  - automatic bounds given any vertex-incremental characterization [Shi, 2016]
- **Random generation:** efficient random generation already possible using cycle pointing [Fusy *et al.* 2007] [Iriza *et al.* 2015].

6. bonus: cactus graphs [with Bahrani, 2017]

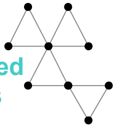
A graph is a **cactus** iff every edge is part of *at most* one cycle.



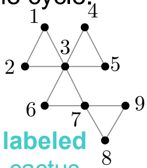
not cactus



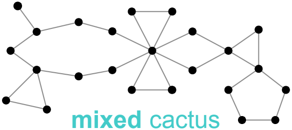
pure 3-cactus



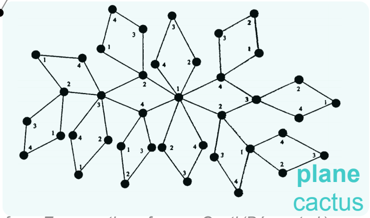
unlabeled cactus



labeled cactus



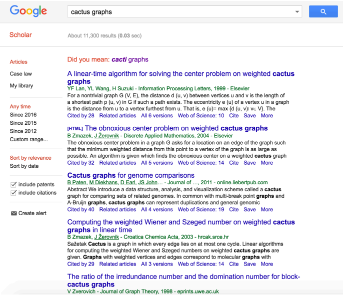
mixed cactus



plane cactus

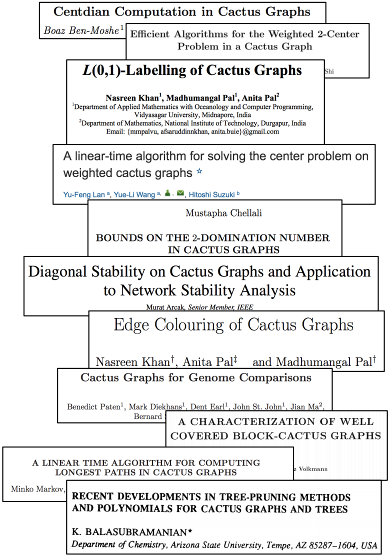
from *Enumeration of m-ary Cacti* (Bóna *et al.*)

prior work on cactus graphs



On the Number of Husimi Trees
Harary and Uhlenbeck (1952):
— derived functional equations for **non-plane, mixed, unlabeled** cacti.

Enumeration of m-ary cacti
Miklós Bóna *et al.* (1999):
— enumerated **pure, plane, unlabeled** cacti.



- cactus graphs are example of split-decomposable graphs with prime nodes that we can characterize
- systematic treatment that can treat plane/non-plane, labeled/unlabeled, pure/mixed
- random generation for all of those graphs