Enumerations Derived from Compact Encodings

Dagstuhl Workshop on Data Structures

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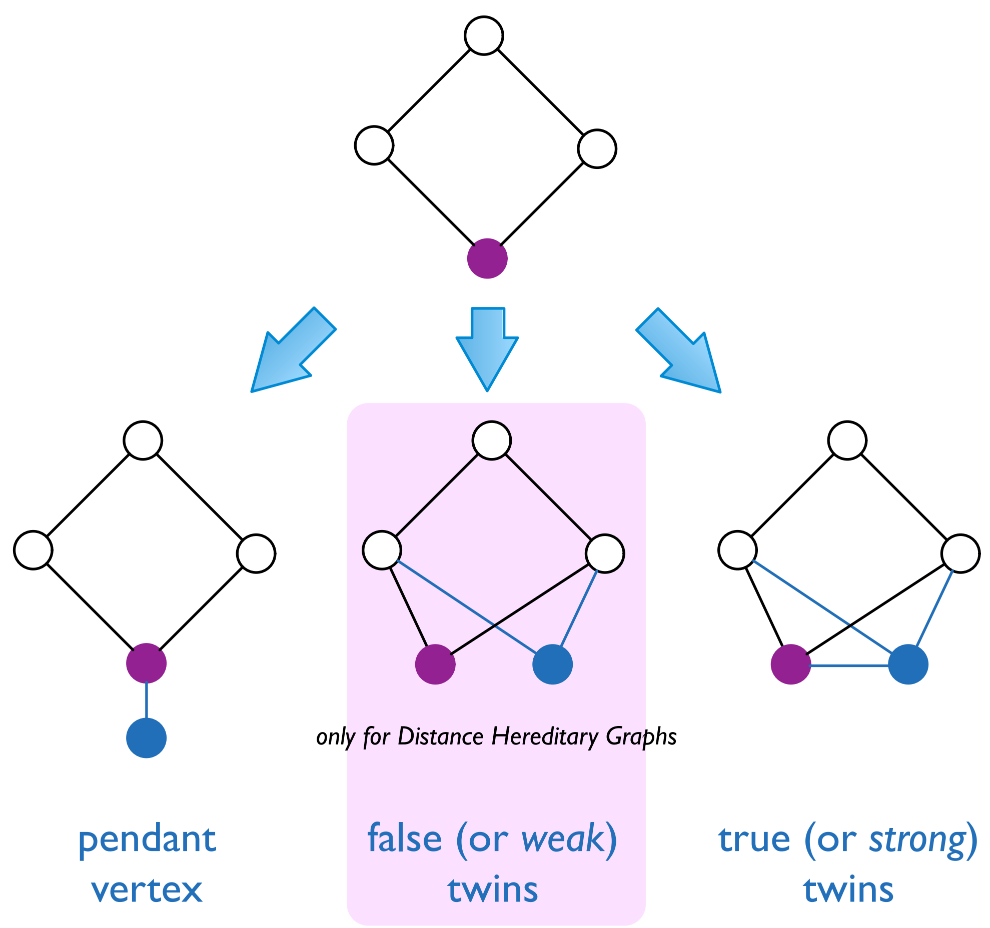
joint work with Jessica Shi (MIT)

Distance Hereditary *vs.* 3-Leaf-Power



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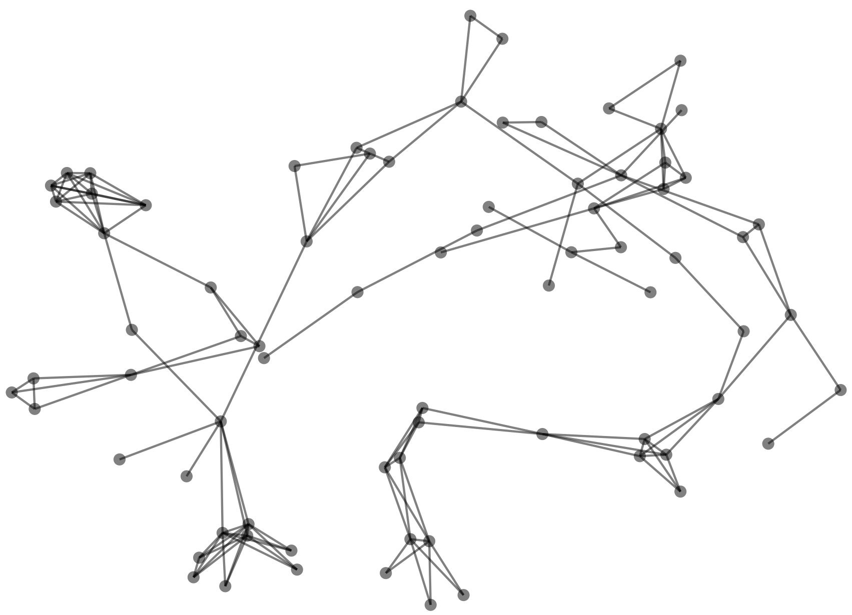
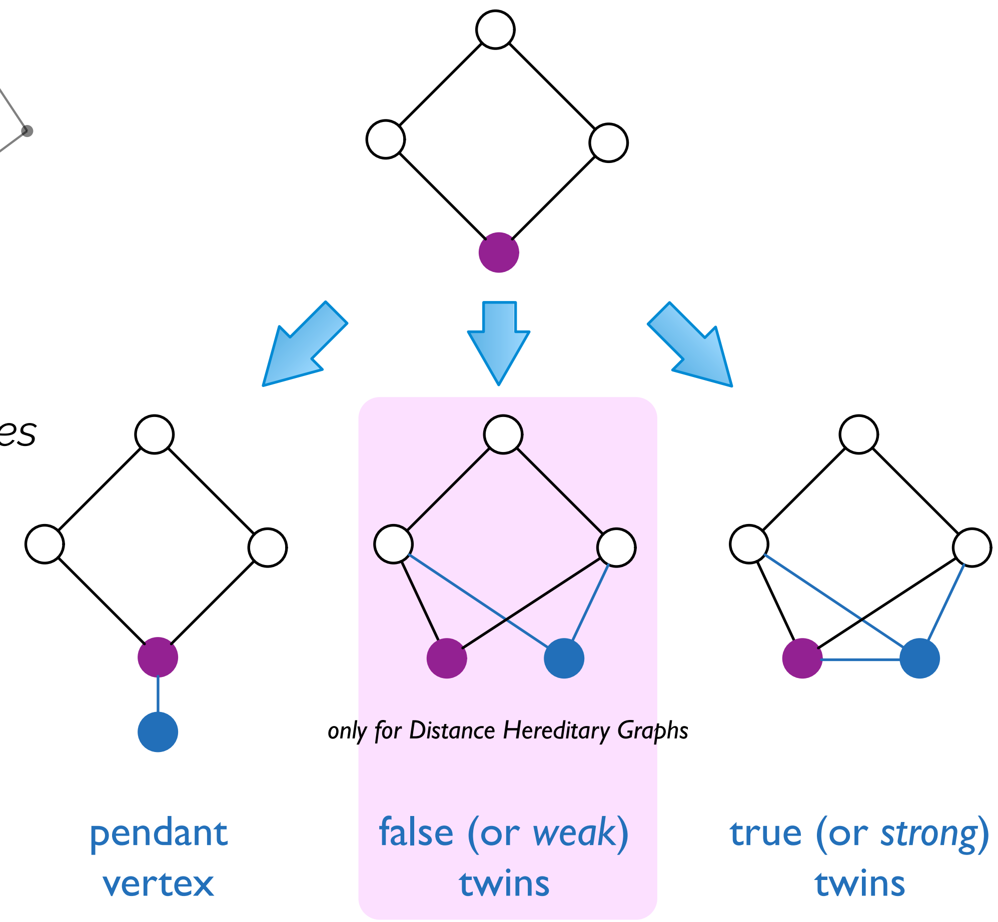


Random unrooted 3LP graph with 77 nodes



All graphs in this slide were plotted by Alex Iriza, 2015.

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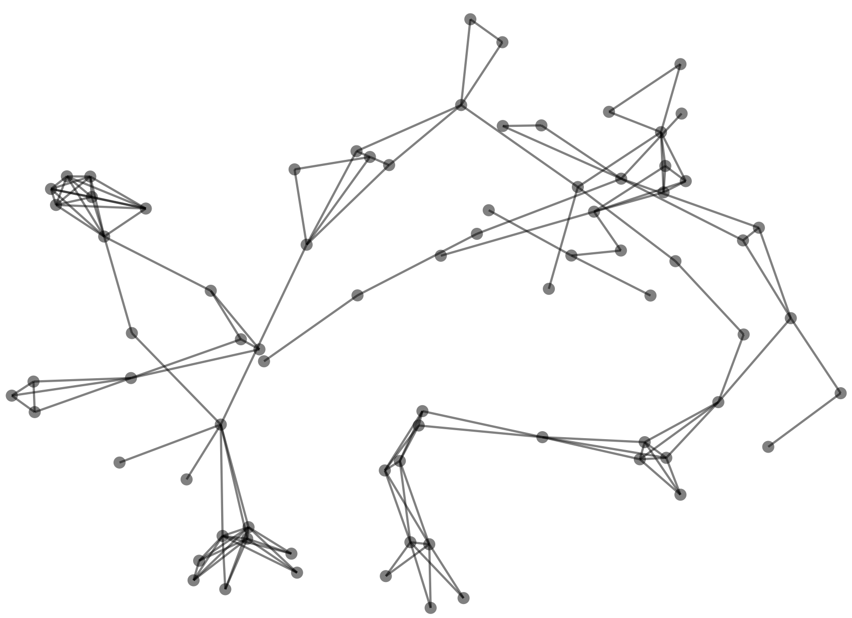

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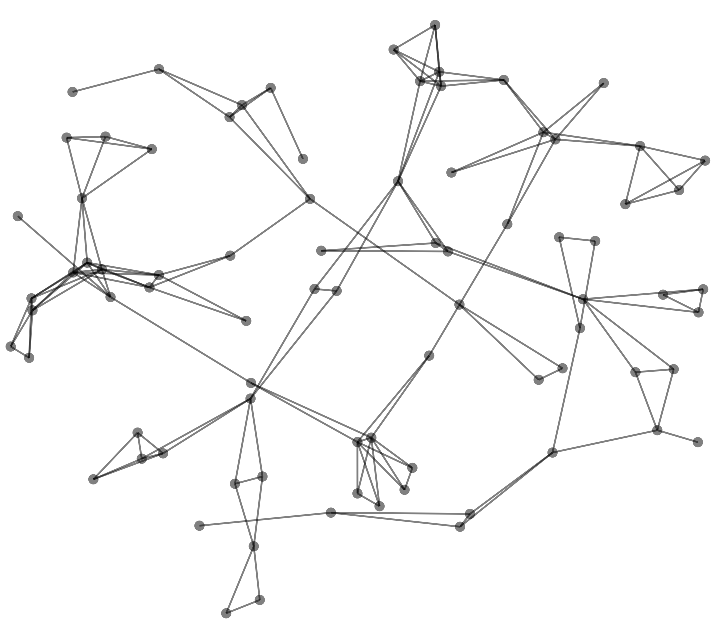
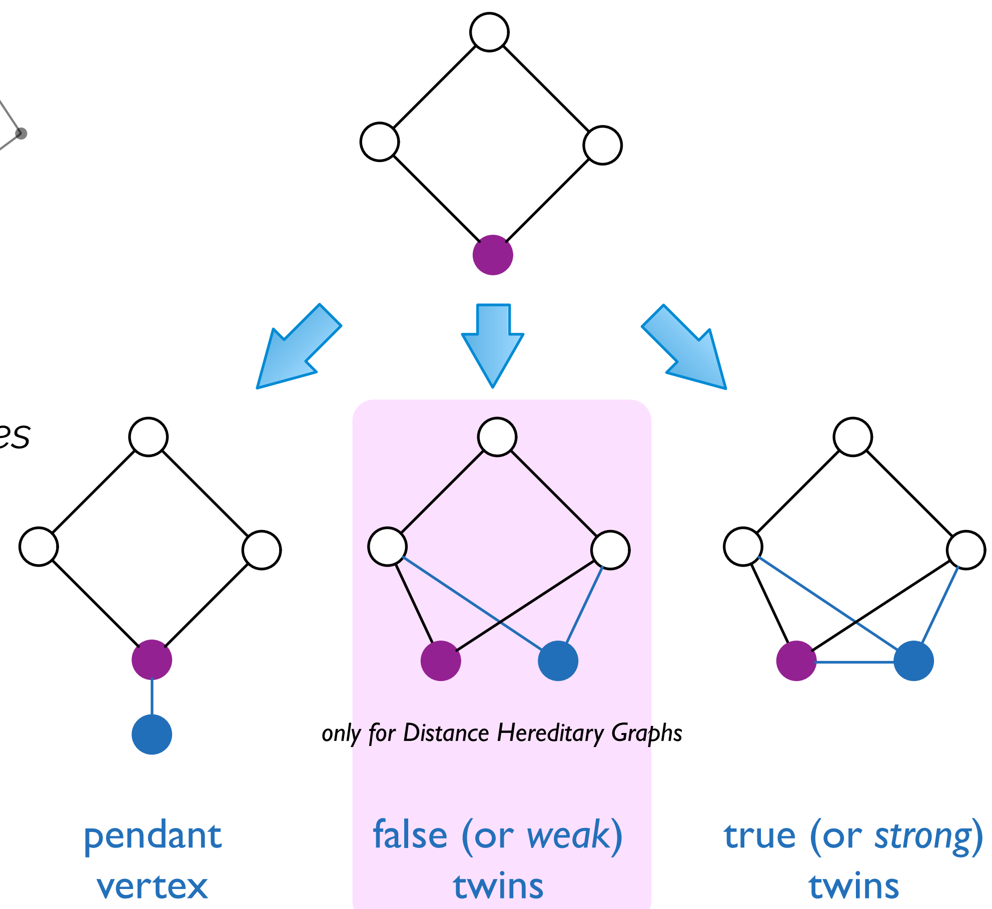


Random unrooted 3LP graph with 82 nodes



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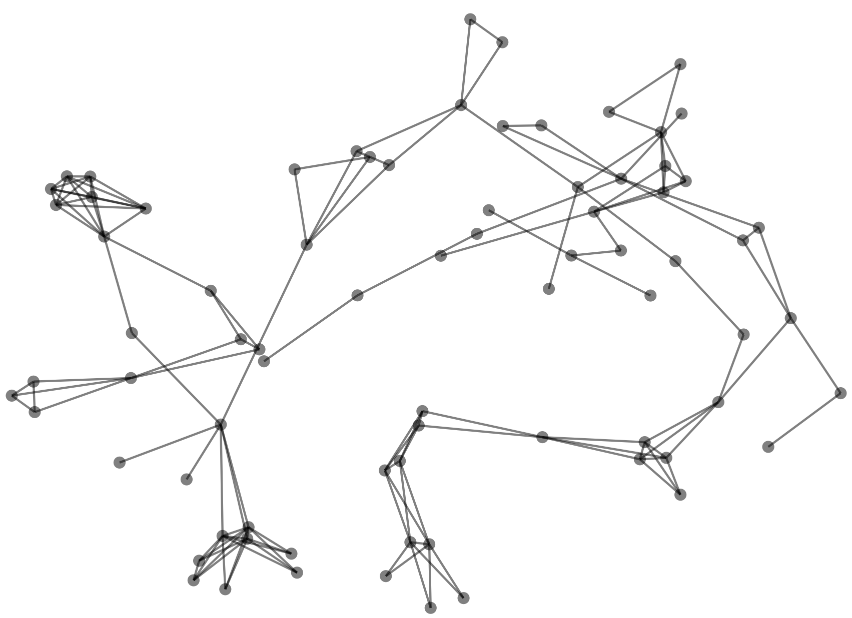

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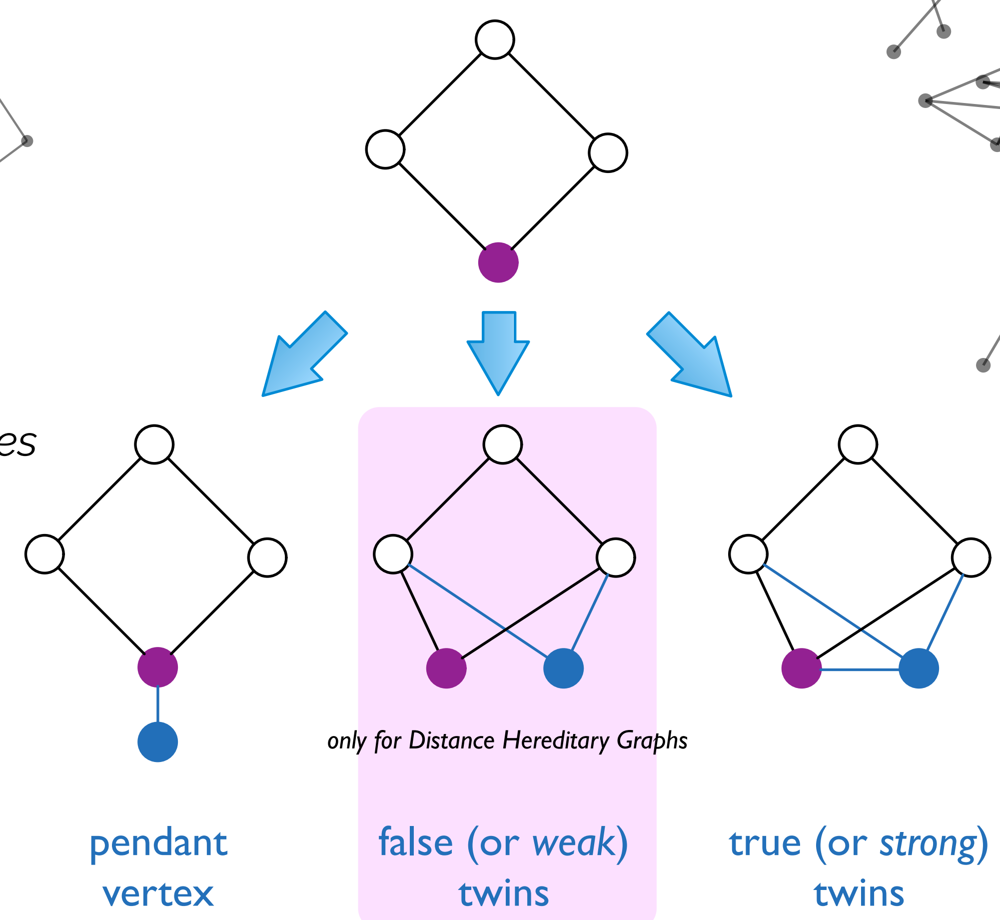
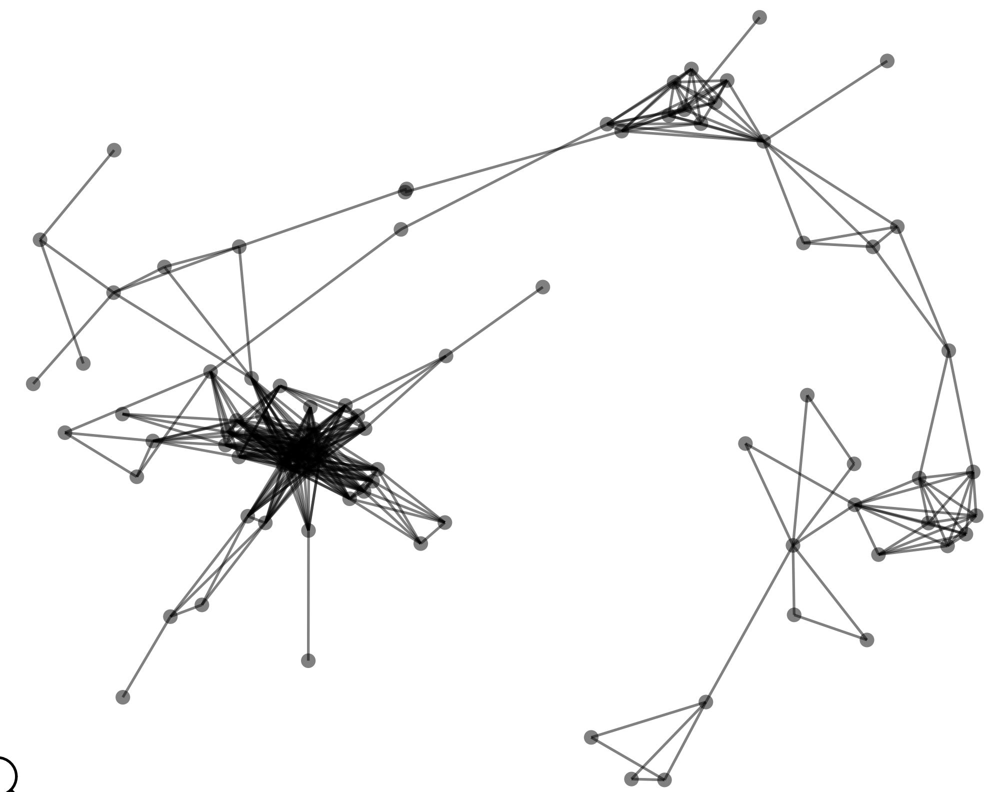
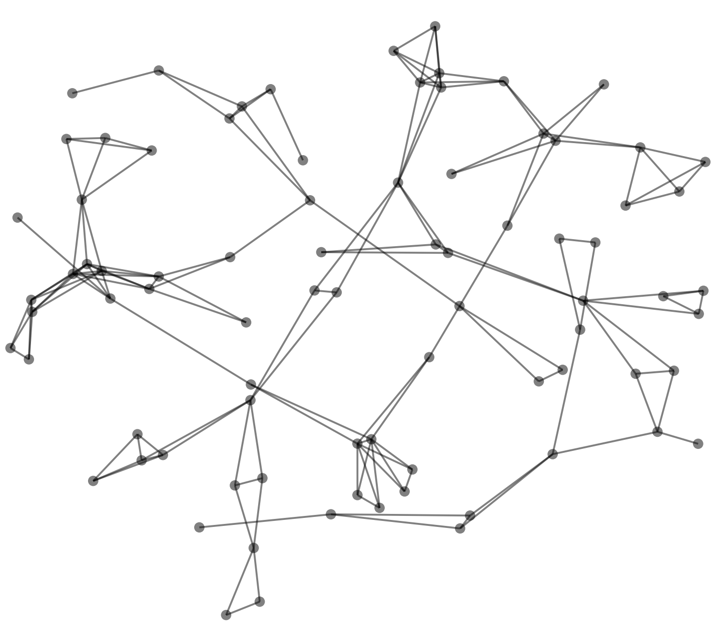




Random unrooted DH graph with 86 nodes



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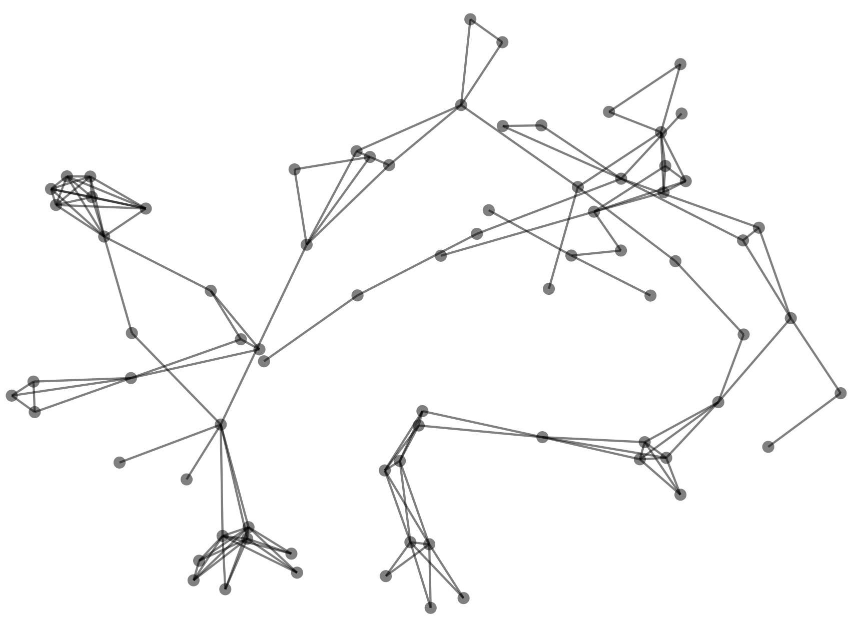

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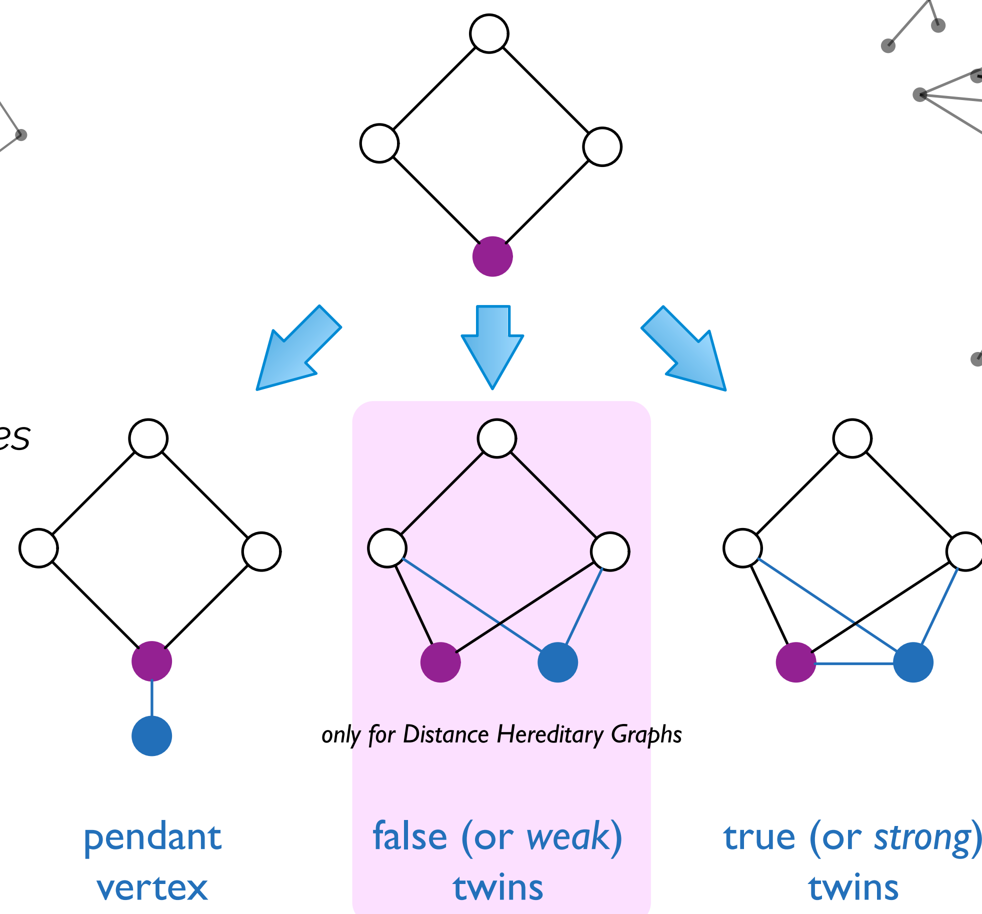

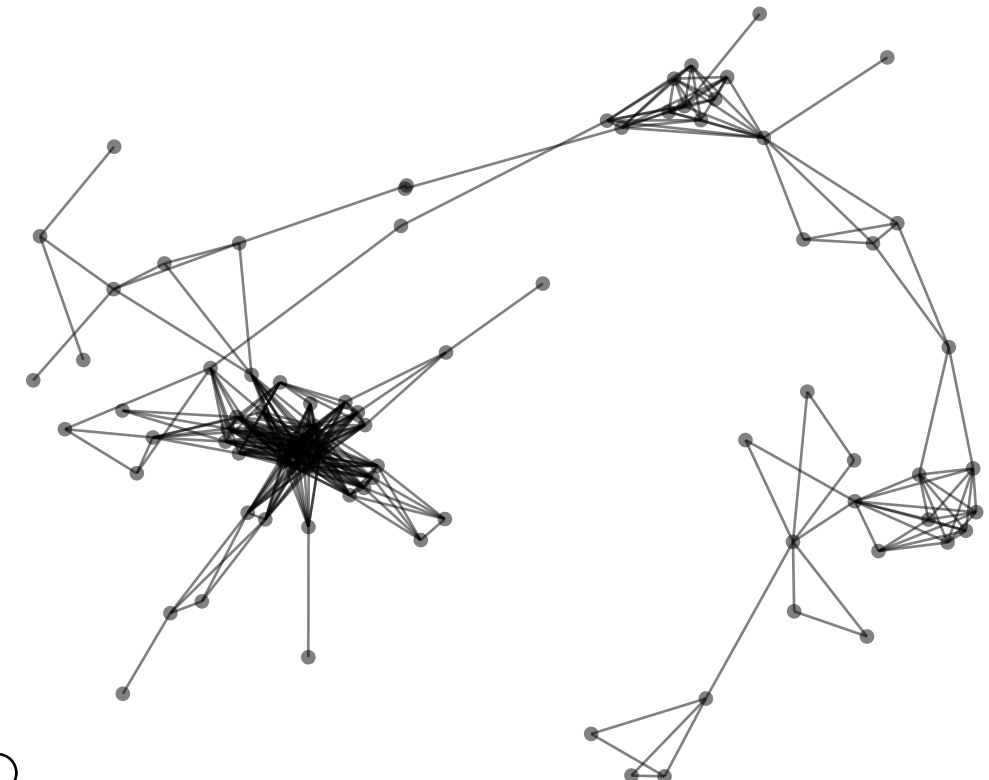

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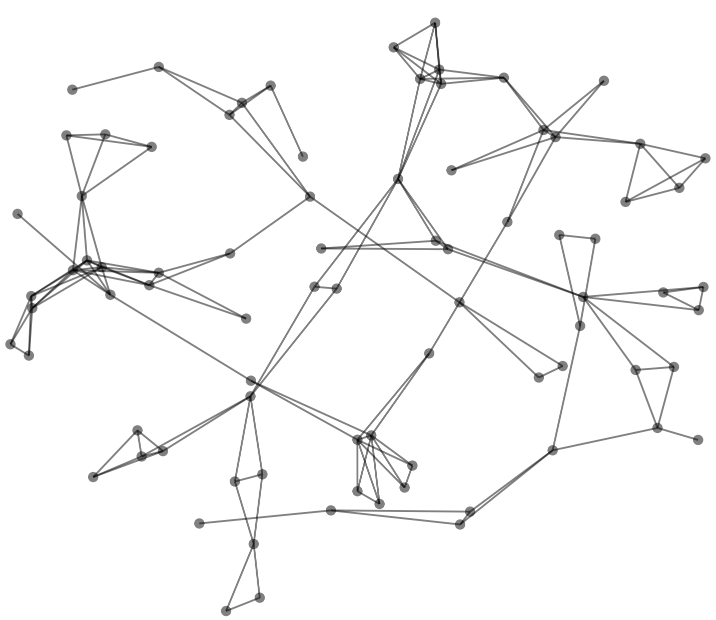
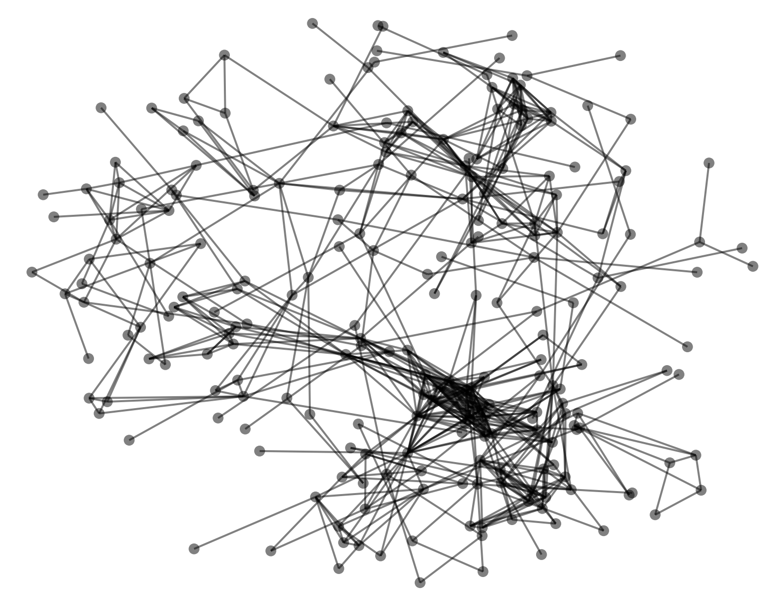


Random unrooted 3LP graph with 82 nodes

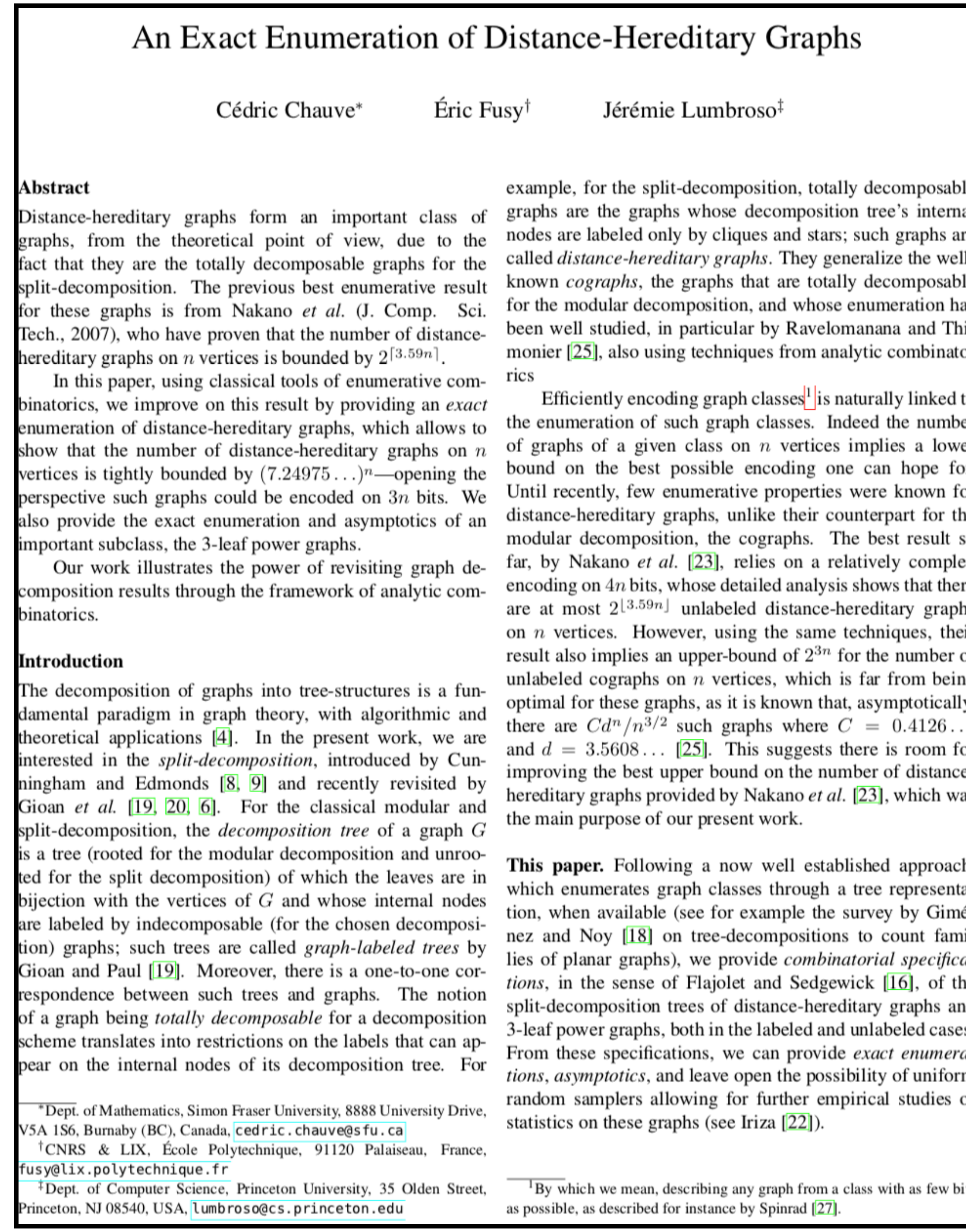


Random unrooted DH graph with 224 nodes

All graphs in this slide were plotted by Alex Iriza, 2015.

Chauve *et al.* methodology (1)

1. **Split-decomposition** of Cunningham, to decompose graphs according to "strongly connected components"
2. Use a tool introduced by Gioan and Paul ("**Graph-labeled trees**") to describe tree
3. Model this tree with **symbolic grammars** (Flajolet & Sedgewick)
4. **Unroot**: Convert from plane to non-plane model with either:
    - A. Dissymmetry Theorem from Species Theory
    - B. Cycle-pointing (more complex)



An Exact Enumeration of Distance-Hereditary Graphs

Cédric Chauve* Éric Fusy† Jérémie Lumbroso‡

Abstract

Distance-hereditary graphs form an important class of graphs, from the theoretical point of view, due to the fact that they are the totally decomposable graphs for the split-decomposition. The previous best enumerative result for these graphs is from Nakano *et al.* (J. Comp. Sci. Tech., 2007), who have proven that the number of distance-hereditary graphs on n vertices is bounded by $2^{\lceil 3.59n \rceil}$.

In this paper, using classical tools of enumerative combinatorics, we improve on this result by providing an *exact* enumeration of distance-hereditary graphs, which allows to show that the number of distance-hereditary graphs on n vertices is tightly bounded by $(7.24975\ldots)^n$—opening the perspective such graphs could be encoded on $3n$ bits. We also provide the exact enumeration and asymptotics of an important subclass, the 3-leaf power graphs.

Our work illustrates the power of revisiting graph decomposition results through the framework of analytic combinatorics.

Introduction

The decomposition of graphs into tree-structures is a fundamental paradigm in graph theory, with algorithmic and theoretical applications [4]. In the present work, we are interested in the *split-decomposition*, introduced by Cunningham and Edmonds [8, 9] and recently revisited by Gioan *et al.* [19, 20, 6]. For the classical modular and split-decomposition, the *decomposition tree* of a graph G is a tree (rooted for the modular decomposition and unrooted for the split decomposition) of which the leaves are in bijection with the vertices of G and whose internal nodes are labeled by indecomposable (for the chosen decomposition) graphs; such trees are called *graph-labeled trees* by Gioan and Paul [19]. Moreover, there is a one-to-one correspondence between such trees and graphs. The notion of a graph being *totally decomposable* for a decomposition scheme translates into restrictions on the labels that can appear on the internal nodes of its decomposition tree. For example, for the split-decomposition, totally decomposable graphs are the graphs whose decomposition tree's internal nodes are labeled only by cliques and stars; such graphs are called *distance-hereditary graphs*. They generalize the well-known *cographs*, the graphs that are totally decomposable for the modular decomposition, and whose enumeration has been well studied, in particular by Ravelomanana and Thimonier [25], also using techniques from analytic combinatorics

Efficiently encoding graph classes[1] is naturally linked to the enumeration of such graph classes. Indeed the number of graphs of a given class on n vertices implies a lower bound on the best possible encoding one can hope for. Until recently, few enumerative properties were known for distance-hereditary graphs, unlike their counterpart for the modular decomposition, the cographs. The best result so far, by Nakano *et al.* [23], relies on a relatively complex encoding on $4n$ bits, whose detailed analysis shows that there are at most $2^{\lfloor 3.59n \rfloor}$ unlabeled distance-hereditary graphs on n vertices. However, using the same techniques, their result also implies an upper-bound of 2^{3n} for the number of unlabeled cographs on n vertices, which is far from being optimal for these graphs, as it is known that, asymptotically, there are $Cd^n/n^{3/2}$ such graphs where $C = 0.4126\ldots$ and $d = 3.5608\ldots$ [25]. This suggests there is room for improving the best upper bound on the number of distance-hereditary graphs provided by Nakano *et al.* [23], which was the main purpose of our present work.

This paper. Following a now well established approach, which enumerates graph classes through a tree representation, when available (see for example the survey by Giménez and Noy [18] on tree-decompositions to count families of planar graphs), we provide *combinatorial specifications*, in the sense of Flajolet and Sedgewick [16], of the split-decomposition trees of distance-hereditary graphs and 3-leaf power graphs, both in the labeled and unlabeled cases. From these specifications, we can provide *exact enumerations*, *asymptotics*, and leave open the possibility of uniform random samplers allowing for further empirical studies of statistics on these graphs (see Iriza [22]).

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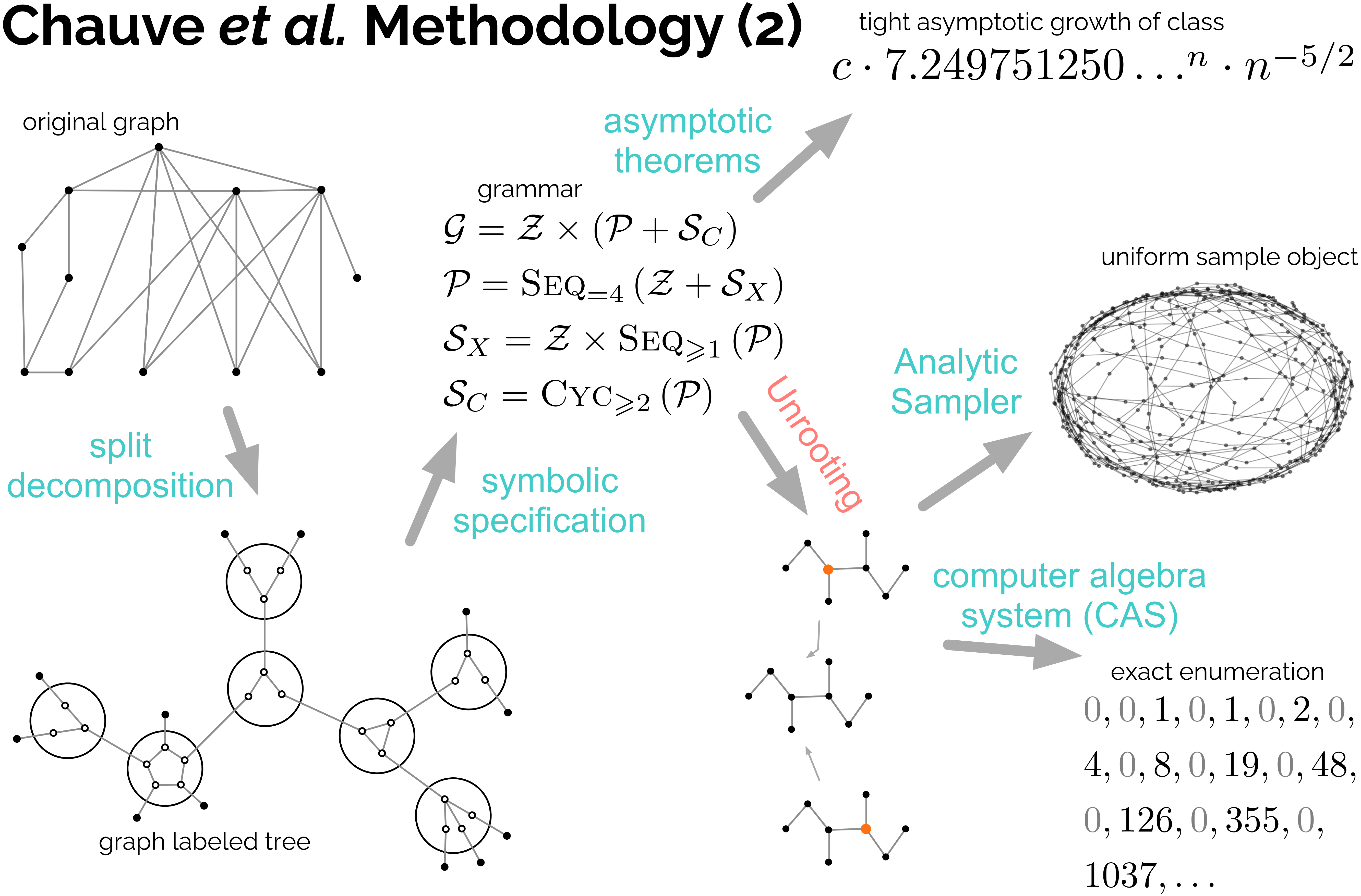
[1] By which we mean, describing any graph from a class with as few bits as possible, as described for instance by Spinrad [27].

Chauve, Fusy, Lumbroso. ICGT 2013, ANALCO 2017.

Chauve *et al.* Methodology (2)

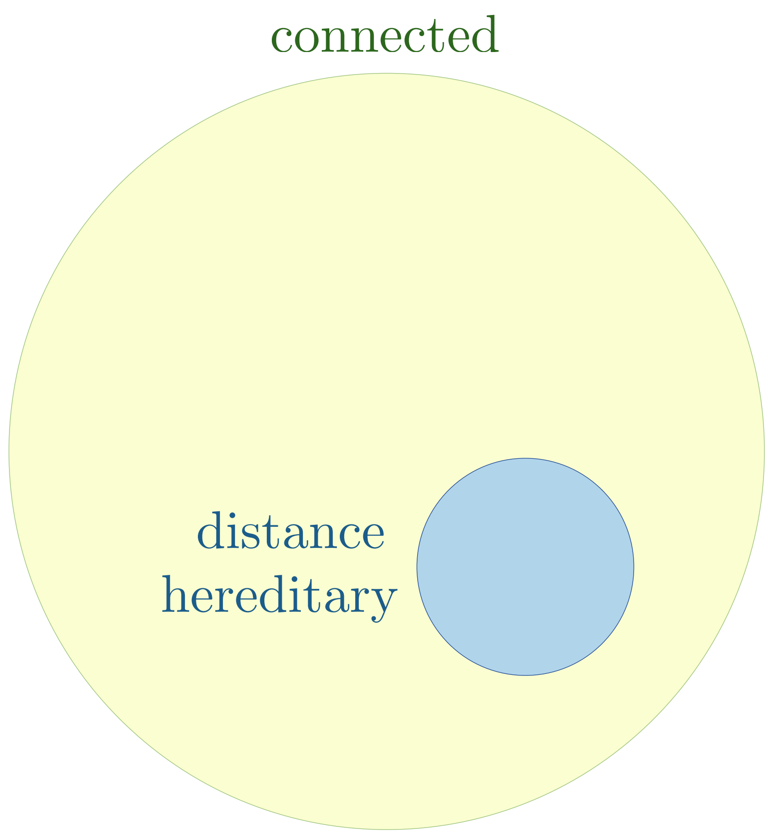


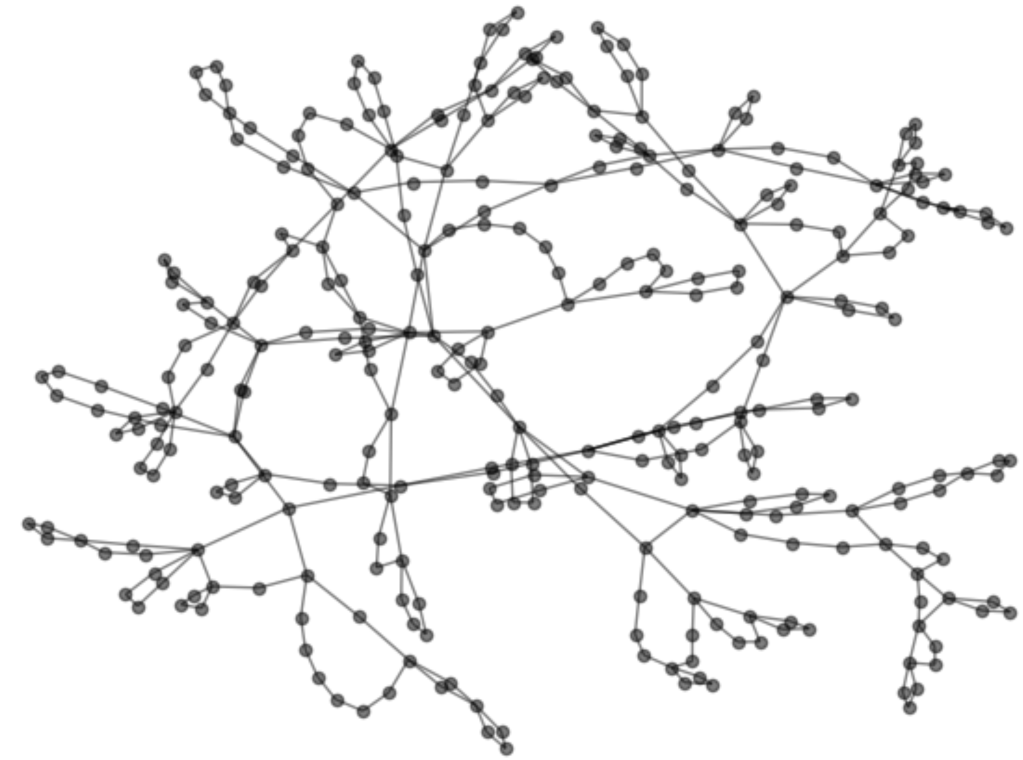
Results

- The methodology has allowed us to recover, for important families of graphs (*in any combination of labeled/unlabeled and rooted/unrooted*):
  - symbolic description
  - exact enumeration
  - efficient random samplers

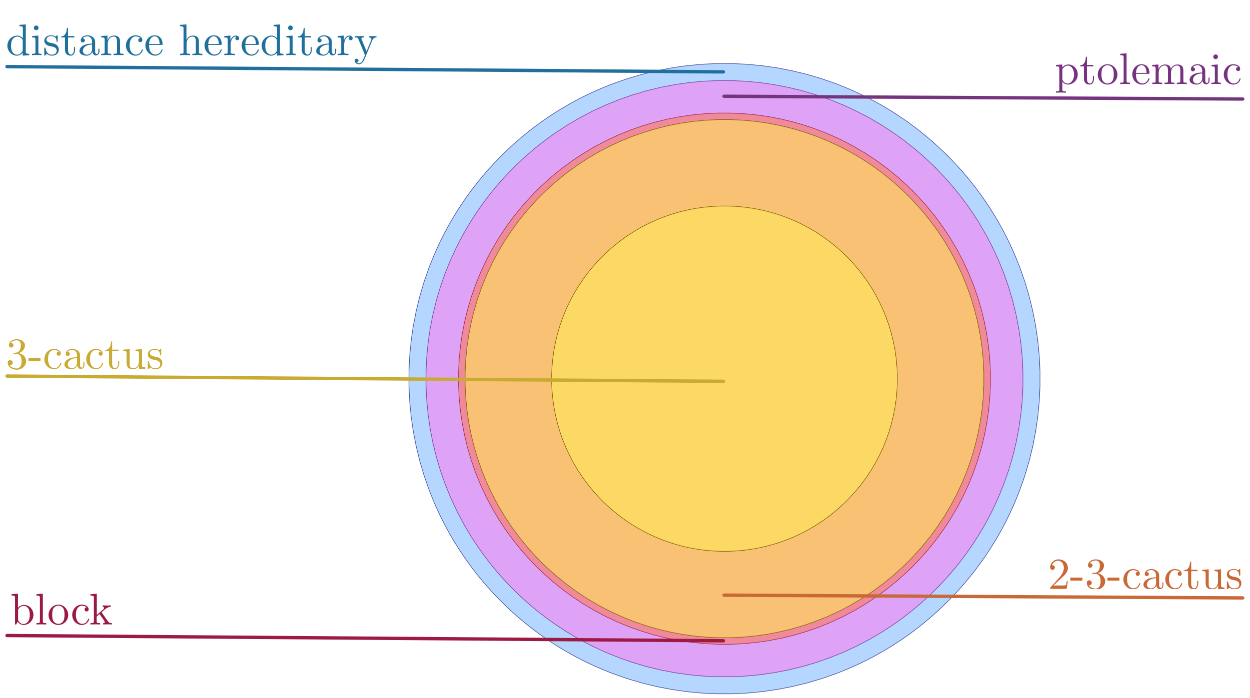
- **Examples** (joint work with Chauve, Fusy, Bahrani, Iriza):
  - *distance-hereditary graphs* (described 1977; exact enumeration 2016);
  - *3-leaf power graphs* (described 2002; exact enumeration 2016);
  - *ptolemaic graphs* [chordal DH] (described 1965; exact enumeration 2017);
  - *cactus graphs* (described 1950, various enumerations discovered since; exact enumeration of all variants 2018);

- Having this information on these graphs makes it drastically easier to make hypotheses, validate and prove them.





A random mixed cactus with
309 vertices and **80 cycles**



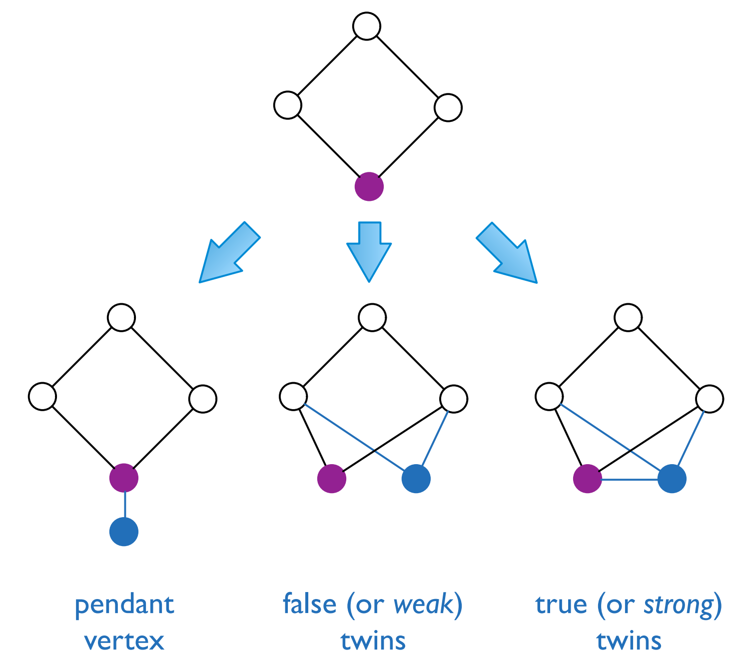
A random mixed cactus with
933 vertices and **239 cycles**

Images by Maryam Bahrani, 2017.

The original inspiration

- Chauve *et al.* (2013, 2017) owes a lot to an article by Nakano *et al.* (2009), with different methodology to get lower-bounds/upper-bounds of distance-hereditary graphs.

- Uses vertex-incremental characterization of DH graphs:
    - `Start from single vertex.`
    - `Repeat until graph has desired size:`
        - `Pick one (or more) vertex, and apply operation`



- **Process to get bounds:**

    1. Describe the sequence of vertex-incremental operation by a tree.

    2. Create constraints on the tree to reduce over-counting.

    3. Describe compact encoding of tree family.

    4. Lower/upper-bound using ad-hoc approximations (*i.e.* "number of bits to store").

- Their result was novel but imprecise.

Nakano S-i, Uehara R, Uno T. A new approach to graph recognition and applications to distance-hereditary graphs. JOURNAL OF COMPUTER SCIENCE AND TECHNOLOGY 24(3): 517–533 May 2009

A New Approach to Graph Recognition and Applications to Distance-Hereditary Graphs*

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Abstract Algorithms used in data mining and bioinformatics have to deal with huge amount of data efficiently. In many applications, the data are supposed to have explicit or implicit structures. To develop efficient algorithms for such data, we have to propose possible structure models and test if the models are feasible. Hence, it is important to make a compact model for structured data, and enumerate all instances efficiently. There are few graph classes besides trees that can be used for a model. In this paper, we investigate distance-hereditary graphs. This class of graphs consists of isometric graphs and hence contains trees and cographs. First, a canonical and compact tree representation of the class is proposed. The tree representation can be constructed in linear time by using prefix trees. Usually, prefix trees are used to maintain a set of strings. In our algorithm, the prefix trees are used to maintain the neighborhood of vertices, which is a new approach unlike the lexicographically breadth-first search used in other studies. Based on the canonical tree representation, efficient algorithms for the distance-hereditary graphs are proposed, including linear time algorithms for graph recognition and graph isomorphism and an efficient enumeration algorithm. An efficient coding for the tree representation is also presented; it requires $\lceil 3.59n \rceil$ bits for a distance-hereditary graph of n vertices and $3n$ bits for a cograph. The results of coding improve previously known upper bounds (both are $2^{O(n \log n)}$) of the number of distance-hereditary graphs and cographs to $2^{\lceil 3.59n \rceil}$ and 2^{3n}, respectively.

Keywords algorithmic graph theory, cograph, distance-hereditary graph, prefix tree, tree representation

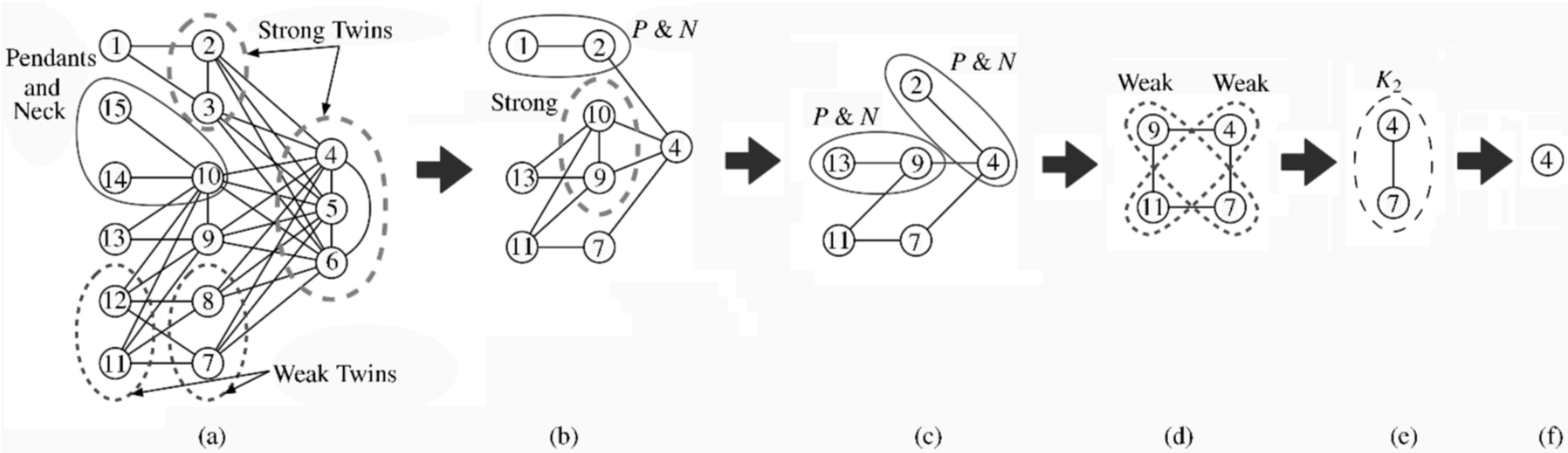


Fig.1. Distance-hereditary graph and its contracting/pruning process.

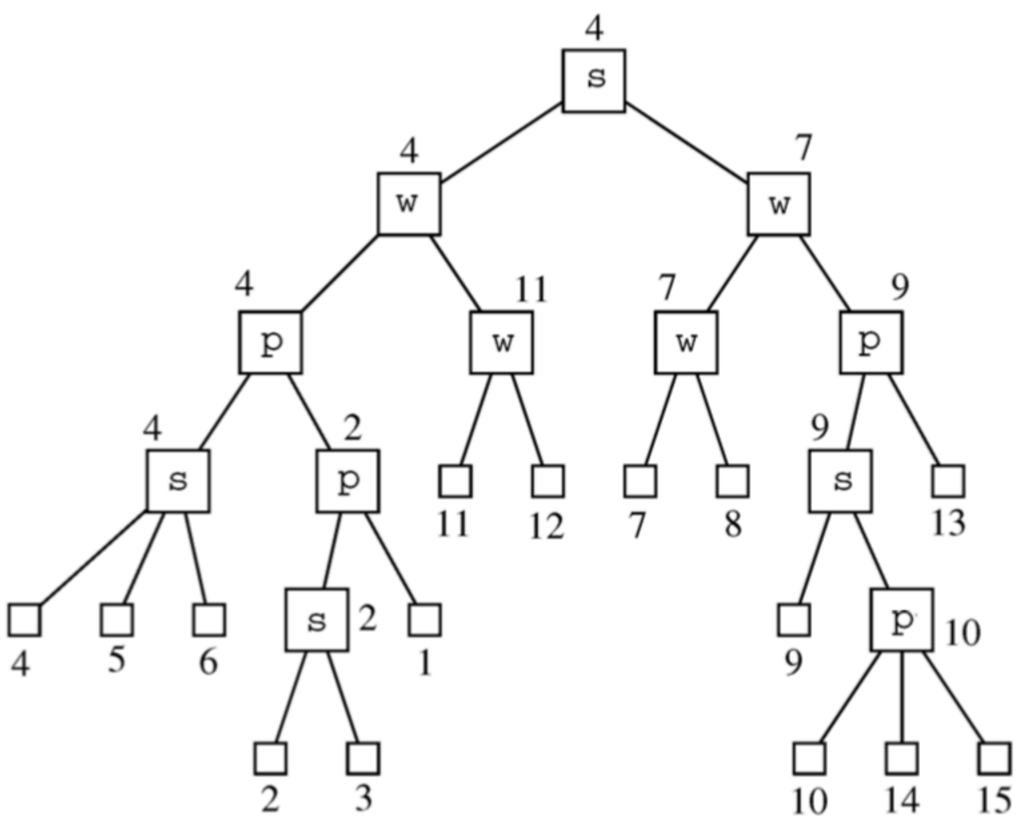


Fig.3. DH-tree $\mathcal{T}$ derived from the graph in Fig.1(a).

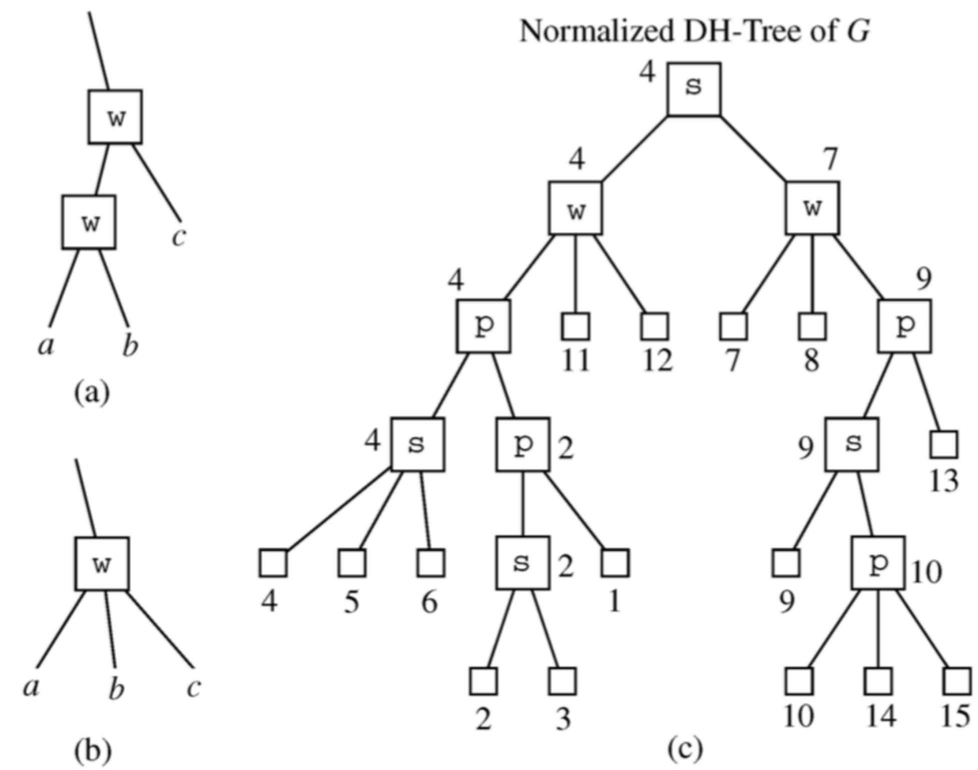
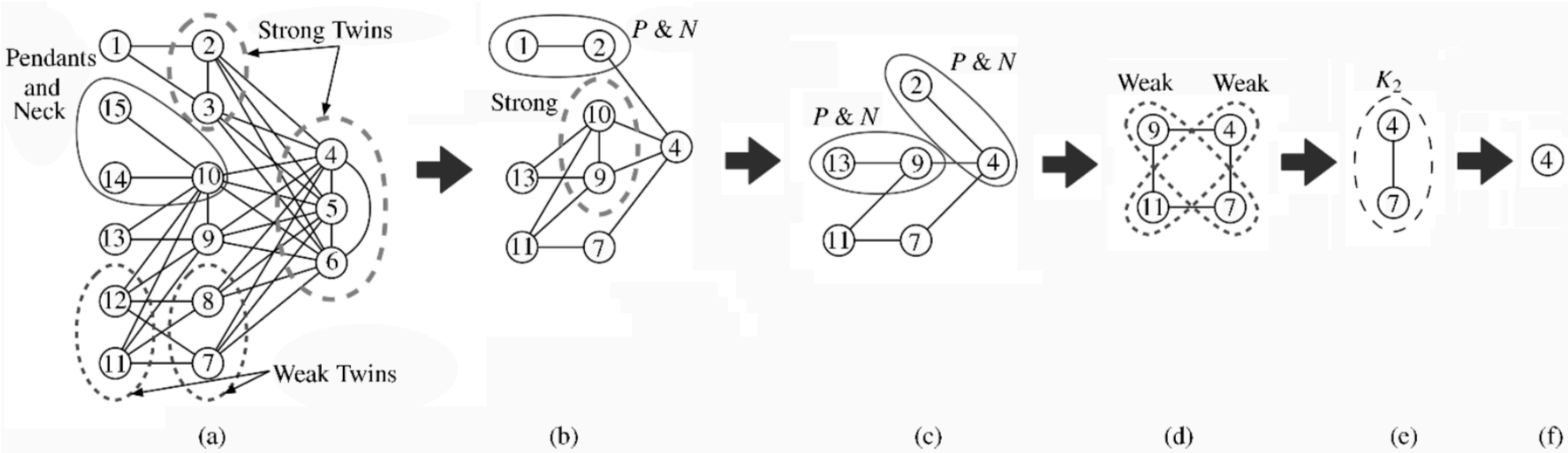


Fig.4. Reduction rule and the compact DH-tree.

$00001\underline{0}1011\underline{0}001\underline{0}11\underline{0}111\underline{0}1011$
$\underline{0}01\underline{0}10001\underline{0}01\underline{0}10111\underline{0}111.$



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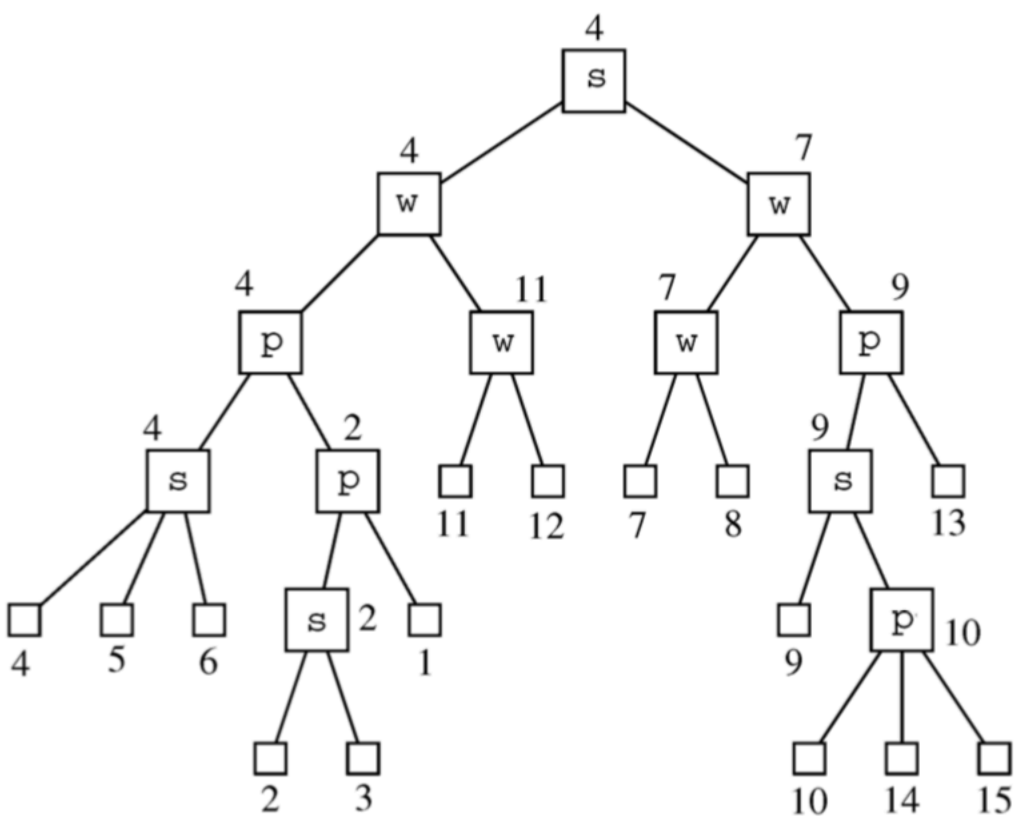

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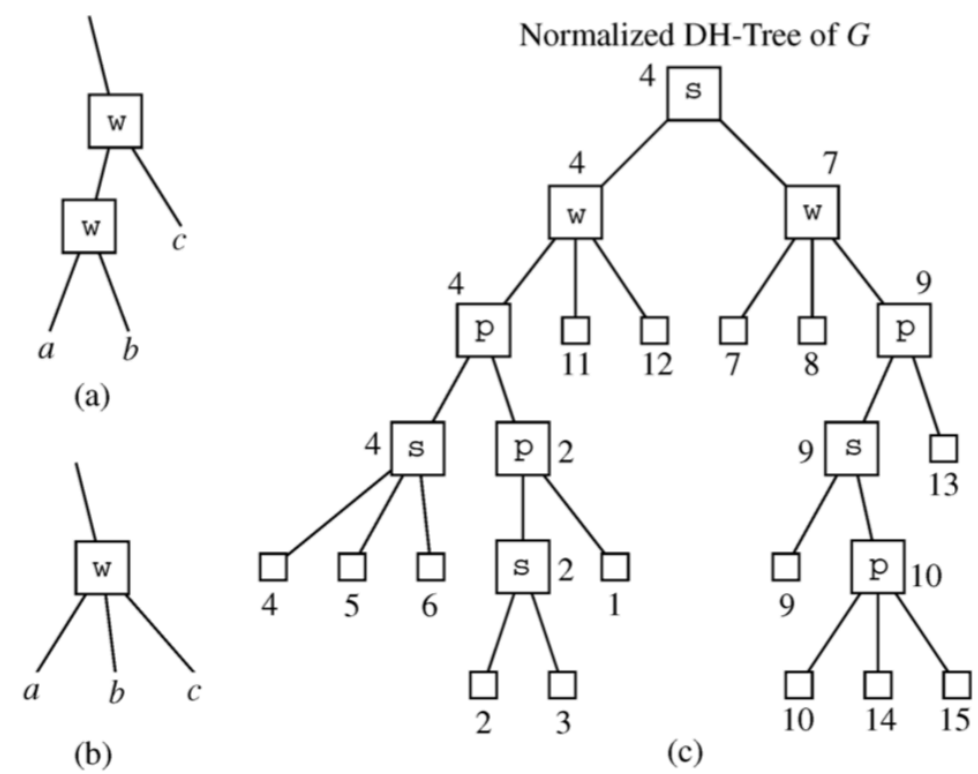

Fig.4. Reduction rule and the compact DH-tree.

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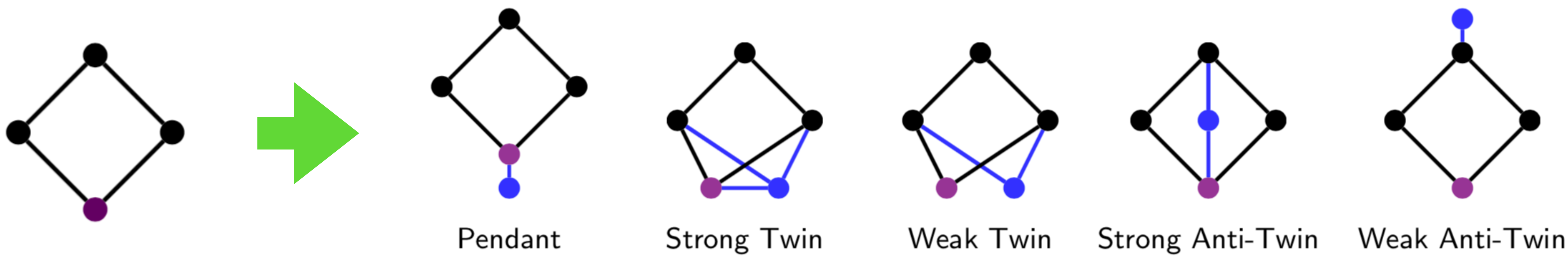
Figures borrowed from Nakano et al. *2009.*

Advantages of Nakano *et al.*

- Although perhaps imprecise, the methodology is flexible and fairly easy to reproduce
- There are many vertex-incremental characterizations (*necessary and sufficient generative conditions*) of various classes of graphs:



Graph Classes	Pendant	Strong twin	Weak twin	Strong anti-twin	Weak anti-twin	Bipartite
3-leaf[3]	1	2				
Cograph[4]		X	X			
Distance-hereditary[5]	X	X	X			
Switch cograph[6]		X	X	X	X	
(6, 2)-chordal bipartite[7]	X		X			
Parity[7]		X	X			X

[3] Gioan and Paul. 2012.
[4] Nakano, Uehara, and Uno. 2009.
[5] Bandelt and Mulder. 1986.
[6] Montgolfier and Rao. 2005.
[7] Cicerone and Di Stefano. 1999.

The Best of Both Worlds?

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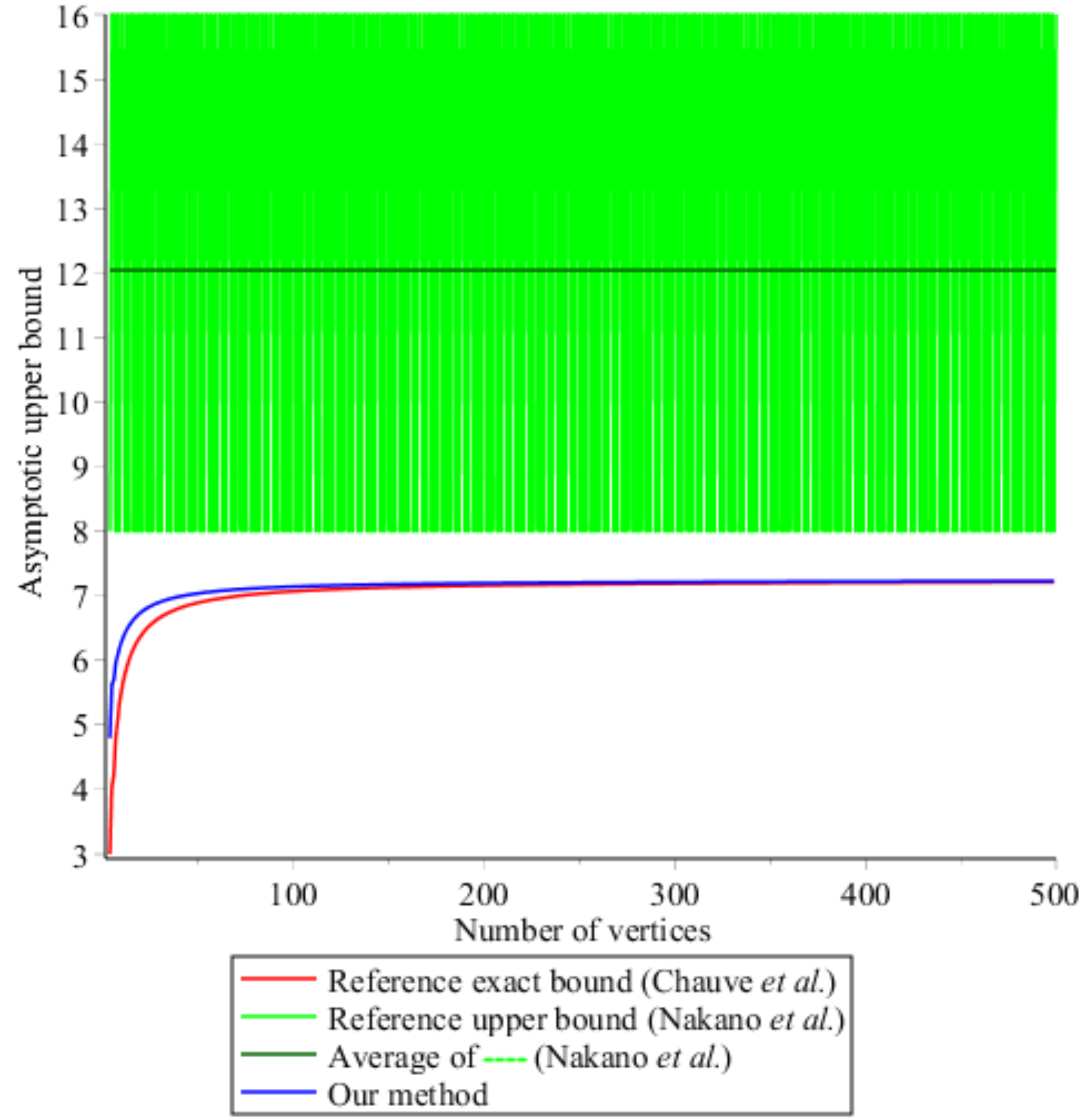
- **Question 2:** More generally, can succinct data structure and/or compact encoding specialists leverage their existing results to obtain more precise enumerations?
- **The answer might be "Yes."**



Plot by Jessica Shi, 2017.

Exponential Bounds from Vertex-Incremental Characterizations

- New methodology

  1. From vertex-incremental characterization, derive tree representation
  2. Find constraints to avoid obvious duplicate trees ("Canonical trees")
  3. Run through black-box analytic combinatorics theorems/CAS

- Quality of bounds depends on rigor of the canonical trees

Exponential Bounds on Graph Enumerations from Vertex Incremental Characterizations

Jérémie Lumbroso* Jessica Shi†

Abstract

In this paper, building on previous work by Nakano *et al.* [23], we develop an alternate technique which almost automatically translates (existing) vertex incremental characterizations of graph classes into asymptotics of that class. Specifically, we construct trees corresponding to the sequences of vertex incremental operations which characterize a graph class, and then use analytic combinatorics to enumerate the trees, giving an upper bound on the graph class. This technique is applicable to a wider set of graph classes compared to the tree decompositions, and we show that this technique produces accurate upper bounds.

We first validate our method by applying it to the case of distance-hereditary graphs, and comparing the bound obtained by our method with that obtained by Nakano et al. [23], and the exact enumeration obtained by Chauve *et al.* [7, 8]. We then illustrate its use by applying it to switch cographs, for which there are few known results: our method provide a bound of $\sim 6.301^n$, to be compared with the precise exponential growth, $\sim 6.159^n$, which we obtained independently through the relationship between switch cographs and bicolored cographs, first introduced by Hertz [19].

We believe the popularity of vertex incremental characterizations might mean this may prove a fairly convenient to tool for future exploration of graph classes.

1 Introduction

Much about trees—their enumeration and asymptotics—is generally well understood; thus, a particularly powerful approach to graph enumeration has been tree decomposition: a bijection establishes a correspondence between a family of graphs with a family of trees, and we study the family of trees. Two well-known examples of such decompositions are the modular decomposition, and the split decomposition [12, 18]. The latter was recently used by Chauve *et al.* [7, 8], to obtain an exact enumeration of an important class of (perfect) graphs, the distance hereditary graphs.

Interestingly Chauve *et al.* built on work by Nakano *et al.* [23], which approximated the distance-hereditary graphs by encoding their construction sequence of operations as a tree; they then used a compact encoding to find a bound for the number of such trees. While this approach proved less accurate—only able to approximate rather than enumerate the distance-hereditary graphs—it's generalization seems to be both directly amenable to tree enumeration and more easily extensible.

The operations in this constructive sequence, are called vertex-incremental operations and they build the graph, by repeated application of any of a (fixed) subset of operation taken from Table 1 to a growing graph starting with a single node. *Vertex incremental* (or *one-vertex extension*) characterizations are the necessary and sufficient conditions under which adding a vertex to a graph in a certain class would produce another graph in that class, and that this operation is generative of the set [4, 24]. [1] A characterization can be viewed algorithmically, as a set of operations under which the class is closed. As such, it is possible to exhaustively enumerate graphs in a certain class using its vertex incremental characterization — this provides us a reference enumeration for small sizes of a graph class. It is also possible to describe the sequence of operations as a tree, to the extent that we need not count the graphs but the combination of operations which builds these graphs (these combinations may provide a superset).

We call these trees *vertex incremental trees* [5], and they are structures that encode the vertex incremental operations used to construct the corresponding graphs. Historically, this idea first emerged in the enumeration of cographs [11]. More recently, Nakano *et al.* [23] used a similar idea for distance-hereditary graphs to obtain an upper bound enumeration. Specifically, Nakano *et al.* used *compact encoding* to enumerate a superset of the vertex incremental trees.

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[1] Like tree decompositions, vertex incremental characterizations have led to algorithmic improvements on certain graph classes. For example, the vertex incremental characterization of distance-hereditary graphs [1] has had applications in obtaining a linear-time algorithm for the domination problem [6] and in deriving linear-time algorithms for weighted vertex cover problems and computing a minimum fill-in and treewidth [5].

Lumbroso, Shi. ANALCO 2018.

- Compared to the exact grammars, the grammars derived from the vertex-incremental methodology are fairly simple
- Requires some patience but no complex math tools

Vertex-incremental bounds (Lumbroso and Shi)

$$\mathcal{DH}_T = \mathcal{PR} + \mathcal{SR} + \mathcal{Z}$$

$$\mathcal{PR} = (\mathcal{S} + \mathcal{W} + \mathcal{Z}) \times \textsc{Set}_{\geq 2}(\mathcal{P} + \mathcal{S} + \mathcal{Z})$$

$$\mathcal{SR} = \textsc{Set}_{\geq 3}(\mathcal{P} + \mathcal{W} + \mathcal{Z}) + \textsc{Set}_{=2}(\mathcal{W}) + \textsc{Set}_{=2}(\mathcal{P}) + \textsc{Set}_{=2}(\mathcal{Z})$$

(Constraints at the root)

$$\mathcal{P} = (\mathcal{S} + \mathcal{W} + \mathcal{Z}) \times \textsc{Set}_{\geq 1}(\mathcal{P} + \mathcal{S} + \mathcal{Z})$$

$$\mathcal{S} = \textsc{Set}_{\geq 2}(\mathcal{P} + \mathcal{W} + \mathcal{Z})$$

$$\mathcal{W} = \textsc{Set}_{\geq 2}(\mathcal{P} + \mathcal{S} + \mathcal{Z})$$

Exact methodology (Chauve *et al.*)

Theorem 4. *The class $\mathcal{DH}$ of unrooted distance-hereditary graphs is specified by*

$$\mathcal{DH} = \mathcal{T}_K + \mathcal{T}_S + \mathcal{T}_{S-S} - \mathcal{T}_{K-S} - \mathcal{T}_{S\to S} \quad (3.25)$$

$$\mathcal{T}_K = \textsc{Set}_{\geqslant 3}(\mathcal{Z} + \mathcal{S}_C + \mathcal{S}_X) \quad (3.26)$$

$$\mathcal{T}_S = (\mathcal{Z} + \mathcal{K} + \mathcal{S}_C) \times \mathcal{S}_C \quad (3.27)$$

$$\mathcal{T}_{K-S} = \mathcal{K} \times (\mathcal{S}_C + \mathcal{S}_X) \quad (3.28)$$

$$\mathcal{T}_{S-S} = \textsc{Set}_2(\mathcal{S}_C) + \textsc{Set}_2(\mathcal{S}_X) \quad (3.29)$$

$$\mathcal{T}_{S\to S} = \mathcal{S}_C \times \mathcal{S}_C + \mathcal{S}_X \times \mathcal{S}_X \quad (3.30)$$

$$\mathcal{K} = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_C + \mathcal{S}_X) \quad (3.31)$$

$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \quad (3.32)$$

$$\mathcal{S}_X = \textsc{Seq}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_C). \quad (3.33)$$

(Derived by describing graph-labeled trees symbolically, and applying the dcissymetry theorem for trees to get the unrooted grammar.)

- Compared to the exact grammars, the grammars derived from the vertex-incremental methodology are fairly simple
- Requires some patience but no complex math tools

Vertex-incremental bounds (Lumbroso and Shi)

$$\mathcal{DH}_T = \mathcal{PR} + \mathcal{SR} + \mathcal{Z}$$

$$\mathcal{PR} = (\mathcal{S} + \mathcal{W} + \mathcal{Z}) \times \textsc{Set}_{\geq 2}(\mathcal{P} + \mathcal{S} + \mathcal{Z})$$

$$\mathcal{SR} = \textsc{Set}_{\geq 3}(\mathcal{P} + \mathcal{W} + \mathcal{Z}) + \textsc{Set}_{=2}(\mathcal{W}) + \textsc{Set}_{=2}(\mathcal{P}) + \textsc{Set}_{=2}(\mathcal{Z})$$

(Constraints at the root)

$$\mathcal{P} = (\mathcal{S} + \mathcal{W} + \mathcal{Z}) \times \textsc{Set}_{\geq 1}(\mathcal{P} + \mathcal{S} + \mathcal{Z})$$

$$\mathcal{S} = \textsc{Set}_{\geq 2}(\mathcal{P} + \mathcal{W} + \mathcal{Z})$$

$$\mathcal{W} = \textsc{Set}_{\geq 2}(\mathcal{P} + \mathcal{S} + \mathcal{Z})$$

Normalization Rules

DH-1. **Commutativity of twins**. The children of a node labeled ${}^wT$ or ${}^sT$ are unordered.

DH-2. **Commutativity of pendants**. The non-leftmost children of a node labeled P are unordered.

DH-3. **Connectivity**. The root is not labeled ${}^wT$.

DH-4. **Associativity of twins**. No child of a node labeled ${}^wT$ can be labeled ${}^wT$, and no child of a node labeled ${}^sT$ can be labeled ${}^sT$.

DH-5. Any non-leftmost child of a node labeled P cannot labeled ${}^wT$.

DH-6. If the root has 2 children, it is labeled ${}^sT$.

DH-7. If the root has 2 children, the labels of the children are either both ${}^wT$ or both P.

DH-8. **Associativity of pendants. The leftmost child of a node labeled P cannot be labeled P.**

Exact methodology (Chauve *et al.*)

Theorem 4. *The class $\mathcal{DH}$ of unrooted distance-hereditary graphs is specified by*

$$\mathcal{DH} = \mathcal{T}_K + \mathcal{T}_S + \mathcal{T}_{S-S} - \mathcal{T}_{K-S} - \mathcal{T}_{S\to S} \quad (3.25)$$

$$\mathcal{T}_K = \textsc{Set}_{\geqslant 3}(\mathcal{Z} + \mathcal{S}_C + \mathcal{S}_X) \quad (3.26)$$

$$\mathcal{T}_S = (\mathcal{Z} + \mathcal{K} + \mathcal{S}_C) \times \mathcal{S}_C \quad (3.27)$$

$$\mathcal{T}_{K-S} = \mathcal{K} \times (\mathcal{S}_C + \mathcal{S}_X) \quad (3.28)$$

$$\mathcal{T}_{S-S} = \textsc{Set}_2(\mathcal{S}_C) + \textsc{Set}_2(\mathcal{S}_X) \quad (3.29)$$

$$\mathcal{T}_{S\to S} = \mathcal{S}_C \times \mathcal{S}_C + \mathcal{S}_X \times \mathcal{S}_X \quad (3.30)$$

$$\mathcal{K} = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_C + \mathcal{S}_X) \quad (3.31)$$

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(Derived by describing graph-labeled trees symbolically, and applying the dcissymetry theorem for trees to get the unrooted grammar.)

Example: Switch cographs

- "Switch cographs" (2005) is a new name for (Bull, Gem, Co-Gem, C5)-free graphs
- No known enumeration or bound
- Vertex-incremental characterization: Strong/weak twin; strong/weak antitoxin

$$\mathcal{SC}_T = \mathcal{ST} + \mathcal{WT} + \mathcal{Z}$$
$$\mathcal{ST} = \textsc{Set}_{\geq 2}(\mathcal{WT} + \mathcal{SA} + \mathcal{Z})$$
$$\mathcal{WT} = \textsc{Set}_{\geq 2}(\mathcal{ST} + \mathcal{WA} + \mathcal{Z})$$
$$\mathcal{SA} = (\mathcal{ST} + \mathcal{WT} + \mathcal{Z}) \times \textsc{Set}_{\geq 1}(\mathcal{ST} + \mathcal{Z})$$
$$\mathcal{WA} = (\mathcal{ST} + \mathcal{WT} + \mathcal{Z}) \times \textsc{Set}_{\geq 1}(\mathcal{WT} + \mathcal{Z})$$

Normalization Rules

SC-1. **Commutativity of twins**. The children of a node labeled ${}^sT$ or ${}^wT$ are unordered.

SC-2. **Commutativity of anti-twins**. The non-leftmost children of a node labeled ${}^s\overline{T}$ or ${}^w\overline{T}$ are unordered.

SC-3. The non-leftmost children of a node labeled ${}^s\overline{T}$ cannot be labeled ${}^wT$. The conjugate is also a normalization.

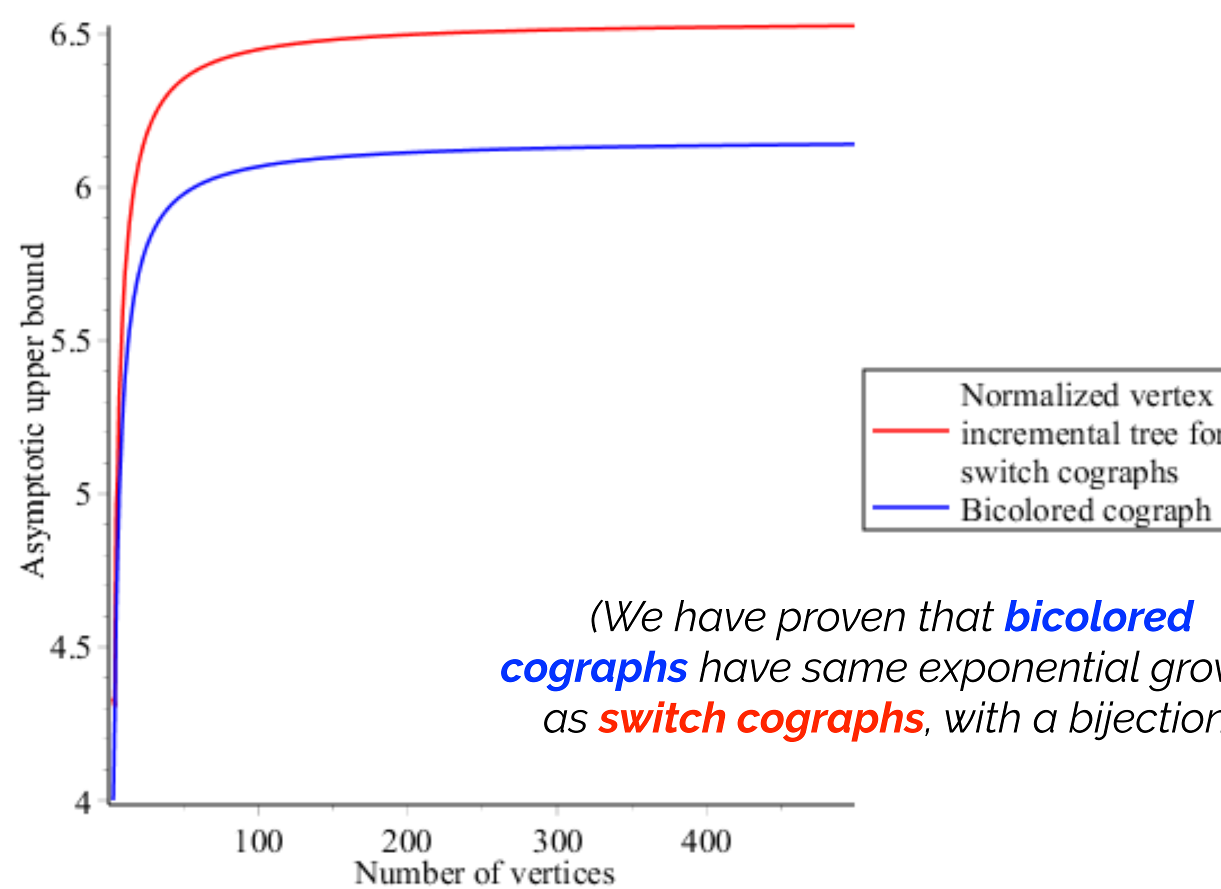
SC-4. The root is not labeled ${}^s\overline{T}$ or ${}^w\overline{T}$.

SC-5. **Associativity of anti-twins**. The children of a node labeled ${}^s\overline{T}$ cannot be labeled ${}^s\overline{T}$. The conjugate is also a normalization.

SC-6. The children of a node labeled ${}^s\overline{T}$ cannot be labeled ${}^w\overline{T}$. The conjugate is also a normalization.

SC-7. **Associativity of twins**. The children of a node labeled ${}^sT$ cannot be labeled ${}^sT$. The conjugate is also a normalization.

SC-8. **Operator associativity of twins and anti-twins**. The children of a node labeled ${}^wT$ cannot be labeled ${}^s\overline{T}$. The conjugate is also a normalization.



*(We have proven that **bicolored cographs** have same exponential growth as **switch cographs**, with a bijection)*

Conclusion

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- Summary of the various methods

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	Extensibility	Accuracy
Nakano *et al.* 2009 *Compact Encoding*	Straightforward	Imprecise bounds (by factor of 2+)
Chauve *et al.* 2013 *Analytic Combinatorics*	Requires familiarity with many mathematical tools	Exact enumeration
Hybrid Method (2018)	Straightforward	Almost tight bounds

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- Interesting to develop methodologies that require less expert-knowledge

- Exploit folklore format description (vertex incremental characterizations, here)